Hamiltonian Structure for Classical Electrodynamics of a Point Particle

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Abstract

We prove that, contrary to the common belief, the classical Maxwell electrodynamics of a point-like particle may be formulated as an infinite-dimensional Hamiltonian system. We derive well defined quasi-local Hamiltonian which possesses direct physical interpretation being equal to the total energy of the composed system (field + particle). The phase space of this system is endowed with an interesting symplectic structure. We prove that this structure is strongly non-degenerated and, therefore, enables one to define the consistent Poisson bracket for particle's and field degrees of freedom. We strees that this formulation is perfectly gauge-invariant.

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1 Introduction

Classical electrodynamics of charged, point-like particles is usually based on the celebrated Lorentz-Dirac equation [1] which, although very useful in many applications, has certain inherent difficulties. There have been many attempts to solve this problem (see e. g. [2] - [5] for the rewiev). However, there is no solution which is accepted by all physicists.

The aim of this paper is to show that despite of these problems the classical electrodynamics of point particles have well defined Hamiltonian structure. We would like to stress that the existence of this structure is nontrivial (up to our knowledge, it is completely unknown).

The derivation of the Lorentz-Dirac equation is based on the decomposition of the electromagnetic field into "retarded" and "incoming" components:

$$f^{\mu\nu} = f^{\mu\nu}_{ret} + f^{\mu\nu}_{in} \ . \tag{1.1}$$

In this approach it is impossible to correctly formulate the initial value problem and find the generator of time translations (see [2]). The dynamics of external "incoming" field is already given and the retarded component is fully determined by the particle's motion. Therefore, the field degrees of freedom are completely eliminated and there is simply no room for the Hamiltonian description. However, recently it was shown (cf. [6]) that classical electrodynamics of point-like objects may be formulated as an infinite-dimensional dynamical system. In this approach particles and field degrees of freedom are kept on the same footing. There is no equations of motion for the particle. It was shown that the conservation law for the total (renormalized) four-momentum is equivalent to a certain boundary condition for the electromagnetic field along the particle's trajectory. Together with this condition, the theory (called *electrodynamics of moving particles*) becomes causal and complete: initial data for both the field and the particles uniquely imply the evolution of the system. This means e. g. that the particles trajectories may also be calculated uniquely from the initial data. Because this approach is relatively little known (and it is crucial for the present paper) we present a short review in the next Section.

It turns out (cf. [7]) that the electrodynamics of moving particles possesses very interesting Lagrangian structure. Let us note that the standard variational principle used in electrodynamics cannot be extended to the theory containing also point-like particles interacting with the electromagnetic field. Such a principle is based on the following Lagrangian, written usually in textbooks:

$$L_{total} = L_{Maxwell} + L_{particle} + L_{int} \ , \tag{1.2}$$

with

$$L_{Maxwell} = -\frac{1}{4}\sqrt{-g}f^{\mu\nu}f_{\mu\nu} \ , \tag{1.3}$$

$$L_{particle} := -m\delta_\zeta \ , \tag{1.4}$$

and the interaction term given by

$$L_{int} := eA_\mu u^\mu \delta_\zeta \ . \tag{1.5}$$

Here by δ_ζ we denote the Dirac delta distribution localized on the particle trajectory ζ. The above Lagrangian may be used to derive the trajectories of the test particles, when the field is given *a priori*. In a different context, it may also be used to derive Maxwell equations, if the particle trajectories are given *a priori*. Simultaneous variation with respect to both fields and particles leads, however, to a contradiction, since the Lorentz force will be always ill defined due to Maxwell equations.

But already in the context of the inhomogeneous Maxwell theory with given point-like sources, the variational principle based on Lagrangian (1.2) is of very limited use, since the interaction term L_{int} becomes infinite. As a consequence, the Hamiltonian of such a theory will always be ill defined, although the theory displays a perfectly causal behaviour.

To improve this bad feature of the theory, a new, quasi-local variational principle for the Maxwell field with given sources was proposed in [7]. This new variational principle is based on the quasi-local Lagrangian which, contrary to (1.2), is well defined, i.e. produces no infinities. It was proved (see [7]) that adding to that Lagrangian the particle Lagrangian (1.4) and varying it with respect to both fields and particles is now possible and does not lead to any contradiction. As a result, one obtains precisely the electrodynamics of moving particles proposed in [6].

In the present paper we give the Hamiltonian formulation of the above theory, i. e. we shall prove that electrodynamics of moving particles may be formulated as an infinite-dimensional Hamiltonian system. For the simplicity we consider here only one particle case. It is of course possible to generalise this result to many particles, however, it is technically much more complicated.

Obviously, the Hamiltonian description based on the standard Lagrangian (1.2) is inconsistent in the case of a point-like sources. In particular, the Hamiltonian of such a theory is always ill defined. Our approach, based on the quasi-local Lagrangian defined in [7] leads *via* an appropriate Legendre transformation to the well defined quasi-local Hamiltonian structure for fields interacting with a charged particle. In particular, the Hamiltonian of the composed "particle + field" system equals numerically to the total energy of the composed system and, therefore, has a direct physical interpretation.

Moreover, it turns out that the above theory possesses highly nontrivial Poisson bracket structure. This structure is defined *via* the symplectic form living in the phase space of the entire system (particle + field). The characteristic feature of the infinite-dimensional symplectic manifold is that the symplectic 2-form is in general only *weakly nondegenerate*, cf. [15]. It means that there need not exist a Hamiltonian vector field $X_\mathcal{F}$ corresponding to every given functional $\mathcal{F}$ on the phase space of the system. For example, our system has well defined Hamiltonian, however, corresponding vector field generated dynamics is not defined throughout the phase space. We prove (cf. Theorem 4) that this vector field is well defined if and only if we restrict the phase space to the states fulfilling the fundamental equation for electrodynamics of moving particles.

Our main result consists in the Theorem 5 which says, that on the reduced phase space the symplectic 2-form becomes *strongly nondegenerate*. This nice mathematical result enables one to define the Poisson bracket structure in the space of functionals over the reduced phase space. To our knowledge the above Poisson bracket structure is the first fully

consistent structure for the theory of interacting particles and fields, i. e. when the particles and fields variables are kept at the same footing.

The paper is organized as follows.

Section 2 contains the main results of the new approach to classical electrodynamics of point particles proposed in [6].

In Section 3 we present a new technique, developed in [7], which enables us to describe at the same footing the field and the particle's degrees of freedom. For this purpose we formulate the Hamiltonian structure of any relativistic, hyperbolic field theory with respect to a non-inertial reference frame defined as a rest-frame for an arbitrarily moving observer.

In Section 4 we show how to extend the above approach to the case of electrodynamical field interacting with point particles.

For the reader convenience we present in Section 5 a new Hamiltonian structure for Maxwell electrodynamics. The new electrodynamical Hamiltonian corresponds to a symmetric energy-momentum tensor and it is related *via* a simple Legendre transformation with electrodynamical Lagrangian derived in [7].

Next three Sections present consistent Hamiltonian structure for electrodynamics of a point particle, i. e. we derive the quasi-local Hamiltonian and find the well defined formula for the Poisson bracket. In Section 9 we show that the above Poisson bracket structure is consistent with Poincaré algebra structure of relativistic theory.

Finally, in Section 10 we present the Hamiltonian formulation for the particle interacting not only with the radiation field, but also with a fixed, external potential, produced by a heavy external device. This is a straightforward extension of our theory, where the homogeneous boundary condition for the radiation field is replaced by an inhomogeneous condition, the inhomogeneity being provided by the external field.

2 The new approach to electrodynamics of a point particle

In the present Section we briefly sketch the new approach to electrodynamics of a point particle presented in [6].

Let $\mathbf{y} = \mathbf{q}(t)$ with $t = y^0$, be the coordinate description of a time-like world line ζ of a charged particle with respect to a laboratory frame, i.e. to a system (y^μ), $\mu = 0, 1, 2, 3;$ of Lorentzian space-time coordinates.

The theory contains as a main part the standard Maxwell equations with point-like sources:

$$\begin{aligned} \partial_{[\lambda} f_{\mu\nu]} &= 0 \ , \\ \partial_\mu f^{\mu\nu} &= e u^\nu \delta_\zeta \ , \end{aligned} \tag{2.1}$$

where u^ν stands for the particle four-velocity and δ_ζ denotes the δ-distribution concentrated on the smooth world line ζ:

$$\delta_\zeta(y^0, y^k) = \sqrt{1 - (\mathbf{v}(y^0))^2} \, \delta^{(3)}(y^k - q^k(y^0)) \ .$$

Here $\mathbf{v} = (v^k) = (\dot{q}^k)$ is the corresponding 3-velocity and $\mathbf{v}^2$ denotes the square of its 3-dimensional length (we use the Heaviside-Lorentz system of units with $c = 1$). In the case of many particles the total current is a sum of contributions corresponding to many disjoint world lines and the value of charge is assigned to each world line separately.

For a given particle's trajectory, equations (2.1) define a deterministic theory. This means that initial data for the electromagnetic field uniquely determine its evolution. However, if we want to treat also the particle's initial data $(\mathbf{q}, \mathbf{v})$ as dynamical variables, the theory based on the Maxwell equations alone is no longer deterministic: the particle's trajectory can be arbitrarily modified in the future or in the past without changing the initial data.

This non-completeness of the theory is usually attributed to the fact that the particle's equations of motion are still missing. However, it was proved in [6] that the field initial data fully determine the particle's acceleration and this is due to Maxwell equations only, without postulating any equations of motion. More precisely, there is a one-to-one correspondence between the (r^{-1})-term of the field in the vicinity of the particle and the acceleration of the particle. The easiest way to describe this property of Maxwell theory is to use the particle's rest-frame. For this purpose consider the 3-dimensional hyperplane Σ_t orthogonal to ζ at the point $(t, \mathbf{q}(t)) \in \zeta$. We shall call Σ_t the "rest frame hyperplane". Choose on Σ_t any system (x^i) of cartesian coordinates centered at the particle's position and denote by r the corresponding radial coordinate. The initial data for the field on Σ_t are given by the electric induction field $\mathbf{D} = (D^i)$ and the magnetic induction field $\mathbf{B} = (B^i)$ fulfilling the conditions $\mathbf{div}\,\mathbf{B} = 0$ and $\mathbf{div}\,\mathbf{D} = e\,\delta_0^{(3)}$. Maxwell equations can be solved for arbitrary data, fulfilling the above constraints, but the solution will be usually non-regular, even far away from the particles. To avoid singularities propagating over a light cone from $(t, \mathbf{q}(t))$, the singular part of the data in the vicinity of the particle has to be equal to

$$D^k = \frac{e}{4\pi} \left[\frac{x^k}{r^3} - \frac{1}{2r}\left(a_i \frac{x^i x^k}{r^2} + a^k \right) \right] + O(1) \,, \tag{2.2}$$

where $\mathbf{a} = (a^k)$ is the acceleration of the particle (in the rest frame we have $a^0 = 0$) and $O(1)$ denotes the nonsingular part of the field (the magnetic field $B^k(r)$ cannot contain any singular part). This gives the one-to-one correspondence between the (r^{-1})-term of the field and the particle's acceleration, which is implied by the regularity of the field outside of the trajectory ζ.

Hence, for regular solutions, the time derivatives $(\dot{\mathbf{D}}, \dot{\mathbf{B}}, \dot{\mathbf{q}}, \dot{\mathbf{v}})$ of the Cauchy data $(\mathbf{D}, \mathbf{B}, \mathbf{q}, \mathbf{v})$ of the composed (fields + particle) system are uniquely determined by the data themselves. Indeed, $\dot{\mathbf{D}}$ and $\dot{\mathbf{B}}$ are given by the Maxwell equations, $\dot{\mathbf{q}} = \mathbf{v}$ and $\dot{\mathbf{v}}$ may be uniquely calculated from equation (2.2). Nevertheless, the theory is not complete and its evolution is not determined by the initial data. This non-completeness may be interpreted as follows. Field evolution takes place not in the entire Minkowski space M, but only outside the particle, i. e. in a manifold with a non-trivial boundary $M_\zeta := M - \{\zeta\}$. The boundary conditions for the field are still missing.

To find this missing condition, the following method was used. Guided by an extended particle model, an "already renormalized" formula was proposed in [6], which assigns to each point $(t, q^k(t))$ of the trajectory a four-vector $p^\lambda(t)$ according to

DEFINITION 1

$$p_\lambda(t) := mu_\lambda(t) + P \int_\Sigma (T^\mu{}_\lambda - \mathbf{T}_{(t)}{}^\mu{}_\lambda)\, d\Sigma_\mu \ , \tag{2.3}$$

where $T^\mu{}_\nu$ denotes the symmetric energy-momentum tensor of the Maxwell field and $\mathbf{T}_{(t)}{}^\mu{}_\nu$ denotes the energy-momentum tensor corresponding to the electromagnetic field produced by the uniformly moving particle along the stright line tangent to the trajectory ζ at $(t, \mathbf{q}(t))$. "P" denotes the principal value of the singular integral and Σ is an arbitrary space-like hypersurface passing through the point $(t, \mathbf{q}(t))$.

It was proved in [6] that the four-vector (2.3) is well defined and does not depend on the particular choice of Σ. It is interpreted as the total four-momentum of the physical system composed of both the particle and the field. For a generic trajectory ζ and a generic solution of Maxwell equations (2.1) this quantity is not conserved, i. e. it depends upon t. The conservation law

$$\frac{d}{dt}\, p^\lambda(t) = 0 \tag{2.4}$$

is proposed as an additional equation, which completes the theory. It was shown that, due to Maxwell equations, only 3 among the 4 equations (2.4) are independent. Given a laboratory reference frame, one may take e. g. the conservation of the momentum $\mathbf{p} = (p^k)$:

$$\frac{d}{dt}\, \mathbf{p} = 0 \tag{2.5}$$

as independent equations. They already imply the energy conservation $\frac{d}{dt}\, p^0(t) = 0$. In Section 6 we will show, that the above momentum $\mathbf{p}$ is equal to the momentum canonically conjugate to the position of the particle, whereas p_0 is equal to the total Hamiltonian of the composed (particle + field) system.

It has been proved in [6] that, due to Maxwell equations, the integral (global) condition (2.5) is equivalent to a (local) boundary condition for the behaviour of the Maxwell field in the vicinity of the trajectory. The condition was called *the fundamental equation* of the electrodynamics of moving particles. In particle's reference frame it may be formulated as a relation between the (r^{-1}) and the (r^0) terms in the expansion of the radial component of the electric field in the vicinity of the particle:

$$D^r(r) = \frac{1}{4\pi}\left(\frac{e}{r^2} + \frac{\alpha}{r}\right) + \beta + O(r)\ , \tag{2.6}$$

where by $O(r)$ we denote terms vanishing for $r \to 0$ like r or faster. For a given value of r both sides of (2.6) are functions of the angles (only the r^{-2} term is angle–independent). The relation between the acceleration and the (r^{-1}) – term of the electric field given in (2.2) may be rewritten in terms of the component α of this expansion:

$$\alpha = -ea_i \frac{x^i}{r} \tag{2.7}$$

(it implies that the quadrupole and the higher harmonics of α must vanish for regular solutions). One can prove the following theorem (see [6]):

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THEOREM 1 *The conservation law (2.5) is equivalent to the following boundary condition*

$$DP(m\alpha + e^2\beta) = 0 \ , \tag{2.8}$$

where DP(f) denotes the dipole part of the function f on the sphere S^2.

The main result of [6] may be summarised in the following

THEOREM 2 *Maxwell equations together with the boundary condition (2.8) define complete, causal and fully deterministic theory: initial data for particles and fields uniquely determine the entire history of the system.*

3 Hamiltonian description of a relativistic field theory in the co-moving frame

3.1 Definition of the co-moving frame

To construct the Hamiltonian formulation of the above theory, we will need a Hamiltonian description of electrodynamics with respect to the particle's rest-frame. In [7] it was shown how to extend the standard variational formulation of field theory to the case of non-inertial frames (for the definition of the co-moving frame see also [8]). In the present Section we show how to extend the standard Hamiltonian formulation of a classical field theory.

Consider any field theory based on a first-order relativistically-invariant Lagrangian density

$$L = L(\psi, \partial\psi) \ , \tag{3.1}$$

where ψ is a (possibly multi-index) field variable, which we do not need to specify at the moment. As an example, ψ could denote a scalar, a spinor or a tensor field.

We will describe the above field theory with respect to accelerated reference frames, related with observers moving along arbitrary space-time trajectories. Let ζ be such a (time-like) trajectory, describing the motion of our observer. Let $y^k = q^k(t)$, $k = 1, 2, 3$; be the description of ζ with respect to a laboratory reference frame, i. e. to a system (y^λ), $\lambda = 0, 1, 2, 3$; of Lorentzian space-time coordinates. We will construct an accelerated reference frame, co-moving with ζ. For this purpose let us consider at each point $(t, \mathbf{q}(t)) \in \zeta$ the 3-dimensional hyperplane Σ_t orthogonal to ζ, i.e. orthogonal to the four-velocity vector $U(t) = (u^\mu(t))$:

$$(u^\mu) = (u^0, u^k) := \gamma(1, v^k) \ , \tag{3.2}$$

where $v^k := \dot{q}^k$ and the relativistic factor $\gamma := 1/\sqrt{1 - \mathbf{v}^2}$. We shall call Σ_t the "rest frame surface". Choose on Σ_t any system (x^i) of *cartesian* coordinates, such that the particle is located at its origin (i. e. at the point $x^i = 0$).

Let us consider space-time as a disjoint sum of rest frame surfaces Σ_t, each of them corresponding to a specific value of the coordinate $x^0 := t$ and parameterized by the coordinates (x^i). This way we obtain a system $(x^\alpha) = (x^0, x^k)$ of "co-moving" coordinates in a neighbourhood of ζ. Unfortunately, it is not always a global system because different Σ's may intersect. Nevertheless, we will use it *globally* to describe the evolution of the field ψ from one Σ_t to another. For a hyperbolic field theory, initial data on one Σ_t imply the entire field evolution. We are allowed, therefore, to describe this evolution as a 1-parameter family of field initial data over subsequent Σ's.

Formally, we will proceed as follows. We consider an abstract space-time $\mathbf{M} := T \times \Sigma$ defined as the product of an abstract time axis $T = \mathbf{R}^1$ with an abstract, three dimensional Euclidean space $\Sigma = \mathbf{R}^3$. Given a world-line ζ, we will need an identification of points of $\mathbf{M}$ with points of physical space-time M. Such an identification is not unique because on each Σ_t we have still the freedom of an O(3)-rotation.

Suppose, therefore, that an identification F has been chosen, which is *local* with respect to the observer's trajectory. By locality we mean that, given the position and the velocity of the observer at the time t, the isometry

$$F_{(\mathbf{q}(t), \mathbf{v}(t))} : \Sigma \mapsto \Sigma_t \tag{3.3}$$

is already defined, which maps $0 \in \Sigma$ into the particle position $(t, \mathbf{q}(t)) \in \Sigma_t$.

As an example of such an isometry which is *local* with respect to the trajectory we could take the one obtained as follows. Choose the unique boost transformation relating the laboratory time axis $\partial/\partial y^0$ with the observer's proper time axis U. Next, define the position of the $\partial/\partial x^k$ - axis on Σ_t by transforming the corresponding $\partial/\partial y^k$ – axis of the laboratory frame by the same boost. It is easy to check, that the resulting formula for F reads:

$$
\begin{aligned}
y^0(t, x^l) &:= t + \gamma(t) x^l v_l(t) \ , \\
y^k(t, x^l) &:= q^k(t) + \left(\delta^k_l + \varphi(\mathbf{v}^2) v^k v_l \right) x^l \ .
\end{aligned}
\tag{3.4}
$$

Here, the following function of a real variable has been used:

$$\varphi(\tau) := \frac{1}{\tau} \left(\frac{1}{\sqrt{1-\tau}} - 1 \right) = \frac{1}{\sqrt{1-\tau}(1 + \sqrt{1-\tau})} \ . \tag{3.5}$$

The function is well defined and regular (even analytic) for $\mathbf{v}^2 = \tau < 1$. The operator

$$\gamma^{-2} \mathbf{a}^k_{\ l} := \delta^k_l + \varphi v^k v_l \tag{3.6}$$

acting on rest-frame variables x^l comes from the boost transformation (for simplicity we skip the argument $\mathbf{v}^2$ of the function φ).

Suppose, therefore, that for a given trajectory ζ a *local* isometry (3.3) has been chosen, which defines $F_\zeta : \mathbf{M} \mapsto M$. This mapping is usually not invertible: different points of $\mathbf{M}$ may correspond to the same point of space-time M because different Σ_t's may intersect. It enables us, however, to define the metric tensor on $\mathbf{M}$ as the pull-back $F^*_\zeta g$ of the Minkowski

metric. The components $g_{\alpha\beta}$ of the above metric are defined by the derivatives of F_ζ, i. e. they depend upon the first and the second derivatives of the position $\mathbf{q}(t)$ of our observer.

Because (x^k) are cartesian coordinates on Σ, the space-space components of g are trivial: $g_{ij} = \delta_{ij}$. The only non-trivial components of g are, therefore, the lapse function and the (purely rotational) shift vector:

$$
\begin{aligned}
N &= \frac{1}{\sqrt{-g^{00}}} = \gamma^{-1}(1 + a_i x^i) \ , \\
N_m &= g_{0m} = \gamma^{-1}\epsilon_{mkl}\omega^k x^l \ ,
\end{aligned}
\tag{3.7}
$$

where a^i is the observer's acceleration vector in the co-moving frame. The quantity ω^m is a rotation, which depends upon the coordination of isometries (3.3) between different Σ_t's. Because ω^m depends locally upon the trajectory, it may also be calculated in terms of the velocity and the acceleration of the observer, once the identification (3.3) has been chosen. In the case of example (3.4), it is easy to check that

$$
a^i = \mathbf{a}^i{}_k \dot{v}^k \ ,
\tag{3.8}
$$

$$
\omega_m = \boldsymbol{\omega}_{ml} \dot{v}^l \ ,
\tag{3.9}
$$

where

$$
\boldsymbol{\omega}_{ml} := \gamma\varphi v^k \epsilon_{klm} \ ,
\tag{3.10}
$$

and $\dot{v}^k$ is the observer's acceleration in the laboratory frame.

The metric $F_\zeta^* g$ is degenerate at singular points of the identification map (i.e. where the identification is locally non-invertible because adjacent Σ's intersect, i. e. where $N = 0$), but this degeneration does not produce any difficulties in what follows.

3.2 Lagrango-Hamiltonian

Once we know the metric (3.7) on $\mathbf{M}$, we may rewrite the invariant Lagrangian density L of the field theory under consideration, just as in any other curvilinear system of coordinates. The Lagrangian obtained this way depends on the field ψ, its first derivatives, but also on the observer's position, velocity and acceleration. Variation with respect to ψ produces field equations in the co-moving system (x^α). Due to the relativistic invariance of the theory, variation of the Lagrangian with respect to the observer's position $\mathbf{q}$ *should not* produce independent equations but only conservation laws, implied already by the field equations.

For our purposes we will keep, however, at the same footing the field degrees of freedom ψ and (at the moment, physically irrelevant) observer's degrees of freedom q^k. We are going to perform the complete "Hamiltonization" of this theory, i.e. to pass to the Hamiltonian description both in field and observer's variables.

Let us first perform a partial Legendre transformation, and pass to the Hamiltonian description of the field degrees of freedom, keeping the Lagrangian description of the "mechanical" degrees of freedom. For this purpose we define

$$L_H := L - \Pi\dot{\psi} \ , \tag{3.11}$$

where Π is the momentum canonically conjugate to ψ:

$$\Pi := \frac{\partial L}{\partial \dot{\psi}} \ . \tag{3.12}$$

The function L_H plays the role of a Hamiltonian (with negative sign) for the fields and a Lagrangian for the observer's position $\mathbf{q}$. It is an analog of the *Routhian function* in analytical mechanics (cf. [7]). The Lagrango-Hamiltonian L_H generates the Hamiltonian field evolution with respect to the accelerated frame, when the "mechanical degrees of freedom" q^k are fixed. Due to (3.7), this evolution is a superposition of the following three transformations:

- time-translation in the direction of the local time-axis of the observer,

- boost in the direction of the acceleration a^k of the observer,

- purely spatial O(3)-rotation ω^m.

It is, therefore, obvious that the numerical value of the generator L_H of such an evolution is equal to

$$L_H = -\gamma^{-1}\left(\mathcal{H} + a^k \mathcal{R}_k - \omega^m S_m\right) \ , \tag{3.13}$$

where $\mathcal{H}$ is the rest-frame field energy, $\mathcal{R}_k$ is the rest-frame static moment and S_m is the rest-frame angular momentum, all of them calculated on Σ. The factor γ^{-1} in front of the generator is necessary, because the time $t = x^0$, which we used to parameterize the observer's trajectory, is not the proper time along ζ but the laboratory time.

3.3 Legendre transformation and relativistic invariance

Now, we perform the Legendre transformation also with respect to the observer's degrees of freedom, and find this way the complete Hamiltonian of the entire (observer + field) system. Let us observe that L_H is a 2-nd order Lagrangian in the observer's variable:

$$L_H = L_H(\mathbf{q}, \mathbf{v}, \dot{\mathbf{v}}, \text{fields}) \ , \tag{3.14}$$

(for the Hamiltonian description of a theory arising from a 2-nd order Lagrangian see Appendix A. A general discussion of a Hamiltonian formalism arising from higher order Lagrangians may be found in [16] - [18]). Let p_k and π_k denote the momenta canonically conjugated to q^k and v^k respectively. The phase space of the entire system, parameterized

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by (q^k, v^k, p_k, π_k) in the observer's sector and by (ψ, Π) in the field sector, is endowed with the canonical symplectic 2-form:

$$\Omega := dp_k \wedge dq^k + d\pi_k \wedge dv^k + \Omega^{field} \ , \tag{3.15}$$

which generates the Poisson bracket for any two observables $\mathcal{F}$ and $\mathcal{G}$:

$$\{\mathcal{F}, \mathcal{G}\} \ := \ \left(\frac{\partial \mathcal{F}}{\partial q^k} \frac{\partial \mathcal{G}}{\partial p_k} - \frac{\partial \mathcal{F}}{\partial p_k} \frac{\partial \mathcal{G}}{\partial q^k} \right) + \left(\frac{\partial \mathcal{F}}{\partial v^k} \frac{\partial \mathcal{G}}{\partial \pi_k} - \frac{\partial \mathcal{F}}{\partial \pi_k} \frac{\partial \mathcal{G}}{\partial v^k} \right) + \{\mathcal{F}, \mathcal{G}\}^{field} \ . \tag{3.16}$$

Due to the relativistic invariance of the theory the following observables: $\mathcal{H}$, $\mathcal{R}_k$, S_m and the rest-frame field momentum $\mathcal{P}_k$ generate with respect to (3.16) the Poincaré algebra:

$$
\begin{aligned}
\{\mathcal{H}, \mathcal{P}_k\}^{field} &= 0 \ , \\
\{\mathcal{H}, S_k\}^{field} &= 0 \ , \\
\{\mathcal{H}, \mathcal{R}_k\}^{field} &= -\mathcal{P}_k \ , \\
\{\mathcal{P}_k, \mathcal{P}_l\}^{field} &= 0 \ , \\
\{\mathcal{P}_k, \mathcal{R}_l\}^{field} &= -g_{kl}\mathcal{H} \ , \\
\{\mathcal{P}_k, S_l\}^{field} &= \epsilon_{klm}\mathcal{P}^m \ , \\
\{\mathcal{R}_k, \mathcal{R}_l\}^{field} &= -\epsilon_{klm}S^m \ , \\
\{\mathcal{R}_k, S_l\}^{field} &= \epsilon_{klm}\mathcal{R}^m \ , \\
\{S_k, S_l\}^{field} &= \epsilon_{klm}S^m \ .
\end{aligned}
\tag{3.17}
$$

To perform the Legendre transformation one has to calculate $\dot{q}^k$ and $\dot{v}^k$ in terms of q^k, v^k, p_k and π_k from formulae (cf. Appendix A):

$$p_k = \frac{\partial L_H}{\partial v^k} - \dot{\pi}_k \ , \qquad \pi_k = \frac{\partial L_H}{\partial \dot{v}^k} \ . \tag{3.18}$$

Since L_H is linear in $\dot{v}$ the Legendre transformation is singular and it gives rise to the Hamiltonian theory with constraints (see [12]- [14]). The *primary constraints* follow from (3.18):

$$\phi_k^{(1)} := \pi_k - \frac{\partial L_H}{\partial \dot{v}^k} = \pi_k + \gamma^{-1} \left(\mathbf{a}^l{}_k \mathcal{R}_l - \boldsymbol{\omega}_{mk} S^m \right) \approx 0 \ . \tag{3.19}$$

By the weak equality symbol "$\approx$" we emphasize that the quantity $\phi_k^{(1)}$ is numerically restricted to be zero but does not identically vanish throughout the phase space. This means in particular, that it has nonzero Poisson bracket with the canonical variables. This way the complete Hamiltonian reads

$$\mathbf{H} = p_k v^k + \pi_k \dot{v}^k - L_H = p_k v^k + \gamma^{-1} \mathcal{H} + \dot{v}^k \phi_k^{(1)} \ . \tag{3.20}$$

The observer's acceleration $\dot{\mathbf{v}}$ plays in (3.20) the role of Lagrangian multiplier. To reduce the theory with respect to the constraints (3.19) and to find the "true" degrees of freedom one has to find the complete hierarchy of constraints via so called Dirac-Bergmann procedure (cf. [12] - [14]).

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PROPOSITION 1 *The secondary constraints $\phi_k^{(2)}$ read:*

$$\phi_k^{(2)} := \{\phi_k^{(1)}, \mathbf{H}\} = -p_k + \gamma v_k \mathcal{H} + \gamma^{-2} \mathbf{a}^l{}_k \mathcal{P}_l \approx 0 . \tag{3.21}$$

Let us find the constraint algebra, i.e. "commutation relations" between constraints $\phi_k^{(a)}$.

PROPOSITION 2

$$\{\phi_k^{(a)}, \phi_l^{(b)}\} = 0 , \tag{3.22}$$

for $a, b = 1, 2$ and $k, l = 1, 2, 3$.

It means that $\phi_k^{(1)}$ and $\phi_k^{(2)}$ are so-called *first-class constraints* (see [12] - [14]). One has to continue this algorithm and look for the constraints which are the "conservation laws" for the $\phi_k^{(2)}$, i.e.

$$\phi_k^{(3)} := \{\phi_k^{(2)}, \mathbf{H}\} \approx 0 . \tag{3.23}$$

However, the *tetriary constraints* $\phi_k^{(3)}$ are satisfied identically due to the Poincarè relations (3.18).

This way we show that the hierarchy of constraints ends on the level of secondary constraints and we obtain the Hamiltonian theory with 6 first-class constraints $\phi_k^{(a)}$. Let $\overline{\mathcal{P}}$ denotes the constraint subspace of the phase space $\mathcal{P}$, i.e.

$$\overline{\mathcal{P}} := \{x \in \mathcal{P} \mid \phi_k^{(a)}(x) = 0 \text{ for } a = 1, 2; \ k = 1, 2, 3\} ,$$

and let $e : \overline{\mathcal{P}} \to \mathcal{P}$ be an embedding. Then, due to (3.22), the pull-back $e^*\Omega$ of the symplectic form Ω is degenerate and to each functional $\phi_k^{(a)}$ corresponds a "gauge direction" $X_k^{(a)}$, i.e.

$$X_k^{(a)} \lrcorner \, e^*\Omega = 0 ,$$

such that

$$e_* X_k^{(a)} \lrcorner \, \Omega = \delta \phi_k^{(a)} ,$$

i.e. $X_k^{(a)}$ is a Hamiltonian vector field corresponding to the functional $\phi_k^{(a)}$ ($X_k^{(a)}$ is tangent to $\overline{\mathcal{P}}$ due to (3.22)).

Therefore, our theory is a gauge theory with 6 gauge parameters: $\mathbf{q}$ and $\mathbf{v}$. Canonically conjugated momenta $\mathbf{p}$ and $\boldsymbol{\pi}$ are subjected to constraints $\phi_k^{(a)}$. Thus, reducing the theory with respect to these constraints (i.e. passing to gauge equivalence classes) we end up with the "true" degrees of freedom, namely those describing the field. Fixing the trajectory plays the role of "gauge fixing" and the "evolution equations" of the observer are automatically satisfied if the field equations are satisfied.

This result is an obvious consequence of the relativistic invariance of the theory. Observer's parameters are not true degrees of freedom and can be easily "gauged away".

4 Renormalized electrodynamical Routhian

We would like to apply the formalism presented in the previous Section for the Hamiltonian description of electrodynamics of a point particle. It turns out that the use of the particle's rest-frame simplifies considerably the formulation of the theory. Hence we will use not an arbitrary observer, but the one following exactly the particle's trajectory. As a consequence, the trajectory will no longer be a gauge parameter, but will have an independent, dynamical meaning, as a new degree of freedom of the theory.

Let us observe that starting from the Lagrangian descripton of electrodynamics given by the standard Maxwell Lagrangian, the Legendre transformation (3.11) does not lead to the correct local expression for the field energy (we obtain the "canonical Hamiltonian" which differs from the field energy by a complete divergence). However, in the "field sector" we may already start with the electrodynamical Routhian (Lagrango-Hamiltonian) (3.13). To obtain this generator we take as $\mathcal{H}$, $\mathcal{R}_k$ and S_m the conventional energy, static moment and angular momentum of the electromagnetic field. These quantities are defined as appropriate integrals of the components of the symmetric energy-momentum tensor:

$$T^\mu_{\ \nu} = f^{\mu\lambda} f_{\nu\lambda} - \frac{1}{4} \delta^\mu_{\ \nu} f^{\kappa\lambda} f_{\kappa\lambda} \ .$$

In the Section 5 we construct the Hamiltonian structure of Maxwell electrodynamics which is perfectly suited for the formalism introduced in the previous Section. In particular, the "new Hamiltonian" is positive definite and equals to the field energy, i.e. is defined as an integral of $T^\mu_{\ \nu}$.

When one adds a point particle to the electromagnetic field, then, obviously, the total field energy is not well defined due to the singularity of the particle's Coulomb field. Therefore, one has to perform renormalization. Decomposing the electric induction field on the rest-frame surface Σ_t into the sum

$$\mathbf{D} = \mathbf{D}_0 + \overline{\mathbf{D}}$$

of the Coulomb field $\mathbf{D}_0 = \frac{e\mathbf{r}}{4\pi r^3}$ and the remaining part $\overline{\mathbf{D}}$, we obtain the following formulae for the renormalized rest-frame quantities (see [6] and [7] for details):

$$\mathcal{H} = m + \frac{1}{2}\int_\Sigma (\overline{\mathbf{D}}^2 + \mathbf{B}^2)\, d^3x \ , \tag{4.1}$$

$$\mathcal{P}_l = \int_\Sigma (\overline{\mathbf{D}} \times \mathbf{B})_l\, d^3x + \int_\Sigma (\mathbf{D}_0 \times \mathbf{B})_l\, d^3x \ , \tag{4.2}$$

$$\mathcal{R}_k = \frac{1}{2}\int_\Sigma x_k (\overline{\mathbf{D}}^2 + \mathbf{B}^2)\, d^3x + \int_\Sigma x_k \overline{\mathbf{D}} \mathbf{D}_0\, d^3x \ , \tag{4.3}$$

$$S_m = \int_\Sigma \epsilon_{mkl} x^k (\overline{\mathbf{D}} \times \mathbf{B})^l\, d^3x \ . \tag{4.4}$$

Finally, the renormalized Lagrango-Hamiltonian L_H is defined as follows: replace in (3.13) the exact values, i. e. calculating for the complete field theory, of the quantities $\mathcal{H}$, $\mathcal{R}_k$ and S_m by the above renormalized quantities (4.1), (4.3) and (4.4), containing only the external

Maxwell field. It was shown in [7] that the variational principle applied to the renormalized L_H gives the Euler-Lagrange equations which are equivalent to the fundamental equation (2.8).

5   The new gauge-invariant Hamiltonian structure of Maxwell electrodynamics

In field theory, contrary to the classical mechanics, there is no unique way to represent the field evolution as an infinite-dimensional Hamiltonian system (see [10] and [11]). Each such representation is based on a specific choice of controlling the boundary value of the field, and corresponds to a specific choice of the Hamiltonian. This non-uniqueness is implied by the non-uniqueness of the evolution of the portion of the field, contained in a finite laboratory V. Indeed, the evolution *is not* unique because external devices may influence the field through the open windows of our laboratory. To choose the Hamiltonian uniquely, we have to insulate the laboratory or, at least, to specify the influence of the external world on it. One may easily imagine an unsuccessful insulation, which does not prevent the external field from penetrating the laboratory. From our point of view, an insulation is sufficient if it keeps under control a complete set of field data on the boundary ∂V in such a way, that the field evolution becomes mathematically unique.

For relatively simple theories (e.g. scalar field theory) the Dirichlet problem may be treated as a privileged one among all possible mixed (initial value + boundary value) problems which are well posed. It turns out (see [10]) that the Dirichlet problem leads to the positive definite Hamiltonian. This means that there is a natural way to insulate the laboratory V adiabatically from the external world. But already in electrodynamics (and even more in General Relativity) any attempt to define the field Hamiltonian leads immediately to the question: how do we *really* define our Hamiltonian system?

5.1   Canonical approach

One usually starts with a canonical dynamical formula for electrodynamics (see [9], [11]):

$$\int_V \dot{\mathcal{F}}^{k0}\delta A_k - \dot{A}_k\delta\mathcal{F}^{k0} \;=\; -\delta H_V^{can} + \int_{\partial V} \mathcal{F}^{\nu\perp}\delta A_\nu \;, \tag{5.1}$$

where H_V^{can} is a "canonical Hamiltonian" related via the Legendre transformation to an electrodynamical Lagrangian density $\mathcal{L}$:

$$H_V^{can} = \int_V (\mathcal{F}^{k0}\dot{A}_k - \mathcal{L}) \;. \tag{5.2}$$

We use standard notation: $\mathcal{F}^{\mu\nu}$ denotes the electromagnetic induction tenor-density. The volume V belongs to the hyperplane Σ_t and consists of the exterior of the sphere $S(r_0)$ (by $\perp$ we denote the component orthogonal to the boundary). We stress that the formula (5.1) is true also in the case of nonlinear electrodynamics. For the linear Maxwell theory $\mathcal{L}$ is

the standard Maxwell Lagrangian $L_{Maxwell} = -\frac{1}{4}\mathcal{F}^{\mu\nu}f_{\mu\nu}$, where $f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. In this case the electromagnetic induction tensor-density is given by $\mathcal{F}^{\mu\nu} := \sqrt{-g}g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta}$.

To describe the boundary term it is convenient to use spherical coordinates (ξ^a), $a = 1, 2, 3$; adapted to ∂V. We choose $\xi^3 = r$ as the radial coordinate and (ξ^A), $A = 1, 2$; as angular coordinates: $\xi^1 = \Theta$, $\xi^2 = \varphi$. The Euclidean metric g_{ab} is diagonal:

$$g_{33} = 1\,, \qquad g_{11} = r^2\,, \qquad g_{22} = r^2 \sin^2\Theta\,,$$

and the volume element $\lambda = (\det g_{ab})^{1/2}$ is equal to $r^2 \sin\Theta$. With this notation we have:

$$\int_V \dot{\mathcal{F}}^{k0}\delta A_k - \dot{A}_k\delta\mathcal{F}^{k0} \;=\; -\delta H_V^{can} + \int_{\partial V} \mathcal{F}^{03}\delta A_0 + \mathcal{F}^{B3}\delta A_B\,. \tag{5.3}$$

The formula (5.3) is analogous to the Hamiltonian formula in classical mechanics

$$\dot{p}_k dq^k - \dot{q}^k dp_k = -dH(q, p)\,.$$

But there is also a boundary term in (5.3), typical for field theory. Killing this term by an appropriate choice of boundary conditions is necessary for transforming the field theory into an (infinite dimensional) dynamical system (see [9], [10]). Thus, boundary conditions for $A_0|\partial V$ and $A_B|\partial V$ make the electrodynamics equivalent to the infinite-dimensional Hamiltonian system. From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of Maxwell electrodynamics described above is mathematically well defined, i.e. a mixed Cauchy problem (Cauchy data given on Σ_t and Dirichlet data given on $\partial V \times \mathbf{R}$) has an unique solution (modulo gauge tranformations which reduce to the identity on $\partial V \times \mathbf{R}$).

5.2   The new approach

It turns out that there is another Hamiltonian structure which is also mathematically well defined (see [11], [10]). Moreover, it possesses very nice properties from the physical point of view:

1. the new Hamiltonian corresponds to the field energy obtained from the symmetric energy-momentum tensor, i.e $\frac{1}{2}(D^2 + B^2)$ in the laboratory frame, which, contrary to the canonical Hamiltonian, is positive definite,

2. this structure is perfectly suited for the reduction of the theory with respect to the Hamiltonian constraint $D^k{}_{,k} = 0$, i.e. Gauss law (see [11]).

Let us start from the canonical relation (5.3)

$$\int_V \dot{\mathcal{F}}^{30}\delta A_k + \dot{\mathcal{F}}^{B0}\delta A_B - \dot{A}_k\delta\mathcal{F}^{30} - \dot{A}_B\delta\mathcal{F}^{B0}$$

$$= \; -\delta H_V^{can} + \int_V \dot{\mathcal{F}}^{30}\delta A_3 + \dot{\mathcal{F}}^{B0}\delta A_B - \dot{A}_3\delta\mathcal{F}^{30} - \dot{A}_B\delta\mathcal{F}^{B0}\,. \tag{5.4}$$

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On each sphere $S(r) = \{r = const\}$ the 2-dimensional covector field A_B splits into a sum of the "longitudinal" and the "transversal" part:

$$A_B = u_{,B} + \epsilon_B{}^C v_{,C} \ , \tag{5.5}$$

where the coma denotes partial differentiation and ϵ^{AB} is a sqew-symmetric tensor, such that $\lambda\epsilon^{AB}$ is equal to the Levi-Civita tensor-density (i. e. $\lambda\epsilon^{12} = -\lambda\epsilon^{21} = 1$). The functions u and v are uniquely given by the field A_B up to additive constants on each sphere separately. Inserting this decomposition into (5.4) and integrating by parts we obtain:

$$\int_V (\dot{\mathcal{F}}^{30}\delta A_3 - \dot{A}_3\delta\mathcal{F}^{30}) - (\dot{\mathcal{F}}^{B0}{}_{,B}\,\delta u - \dot{u}\delta\mathcal{F}^{B0}{}_{,B}) - (\dot{\mathcal{F}}^{B0}{}_{,C}\,\epsilon_B{}^C\delta v - \dot{v}\mathcal{F}^{B0}{}_{,C}\,\epsilon_B{}^C)$$

$$= -\delta H_V^{can} + \int_{\partial V} \mathcal{F}^{03}\delta A_0 - \mathcal{F}^{B3}{}_{,B}\,\delta u - \mathcal{F}^{B3}{}_{,C}\,\epsilon_B{}^C\delta v \ . \tag{5.6}$$

Using identities $\partial_B\mathcal{F}^{B0} + \partial_3\mathcal{F}^{30} = 0$ and $\partial_B\mathcal{F}^{B3} + \partial_0\mathcal{F}^{03} = 0$, implied by the field equations $\partial_\mu\mathcal{F}^{\mu\nu} = 0$, and integrating again by parts we finally obtain:

$$\int_V [\dot{\mathcal{F}}^{30}\delta(A_3 - u_{,3}) - (\dot{A}_3 - \dot{u}_{,3})\delta\mathcal{F}^{30}] + [\dot{\mathcal{F}}^{0B||C}\epsilon_{BC}\delta v - \dot{v}\delta\mathcal{F}^{0B||C}\epsilon_{BC}]$$

$$= -\delta H_V^{can} + \int_{\partial V} \mathcal{F}^{03}\delta(A_0 - \dot{u}) + \mathcal{F}^{3B||C}\epsilon_{BC}\delta v \ . \tag{5.7}$$

Here, by "$||$" we denote the 2-dimensional covariant derivative on each sphere $S(r)$. The quantities $(A_0 - u_{,0})$ and $(A_3 - u_{,3})$ are "almost" gauge invariant: only their monopole part (mean-value) on each sphere may be affected if we change the additive constant in the definition of u (the choice of an additive constant in the definition of v is irrelevant, because it is always multiplied by quantities which vanish when integrated over a sphere). The sum of the volume and surface integrals in (5.7) is however gauge invariant. Now,

$$B^A = (\mathbf{curlA})^A = \epsilon^{AB}(A_{B,3} - A_{3,B}) \tag{5.8}$$

and using (5.5) we have

$$\triangle(A_3 - u_{,3}) = r^2 B^{A||B}\epsilon_{AB} \ , \tag{5.9}$$

where $\triangle$ denotes the 2-dimensional Laplace-Beltrami operator on $S(r)$ multiplied by r^2 (the operator $\triangle$ does not depend on r and is equal to the Laplace-Beltrami operator on the unit sphere $S(1)$). The operator $\triangle$ is invertible on the space of monopole–free functions (functions with vanishing mean value on each $S(r)$). This functional space will play an important role in further considerations and all the dynamical field quantities of the theory will belong to this space. To fix both terms in (5.7) uniquely we choose u in such a way that the mean value of $(A_3 - u_{,3})$ vanishes on each sphere. Hence, with the above choice of the additive constants the quantity $A_3 - u_{,3}$ becomes gauge invariant:

$$A_3 - u_{,3} = r^2\triangle^{-1}(B^{A||B}\epsilon_{AB}) \ . \tag{5.10}$$

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Let us observe that the function v is also gauge invariant (up to an additive constant, which does not play any role and may also be chosen in such a way that its mean value vanishes on each sphere). Indeed, we have:

$$B^3 = (\mathbf{curl\,A})^3 = A^{A||B}\epsilon_{BA} = -r^{-2}\triangle v\,. \tag{5.11}$$

Due to the Maxwell equation $\mathbf{div\,B} = 0$, the function B^3 is monopole–free and the Laplasian $\triangle$ may again be inverted:

$$v = -r^2\triangle^{-1}B^3\,. \tag{5.12}$$

The formula (5.7) could be also obtained directly from (5.4) by imposing the following gauge conditions:

$$A^B_{||B} = 0\,, \tag{5.13}$$

$$\int_{S(r)} \lambda A_3 = 0\,. \tag{5.14}$$

The above condition does not fix the gauge uniquely: we still may add to A_μ the gradient of a function of time $f = f(t)$. This residual gauge changes only the monopole part of A_0, but both the volume and the surface integrals in (5.3) remain invariant with respect to such a transformation.

Assuming the above gauge, we have $u \equiv 0$ and $A_B = \epsilon_B{}^C v_{,C}$. To simplify the notation we will, therefore, replace our invariants $(A_0 - u_{,0})$ and $(A_3 - u_{,3})$ by the values of A_0 and A_3, calculated in this particular gauge.

Let us observe that formula (5.7) represents dynamical system with infinitely many degrees of freedom described by four functions: $\mathcal{F}^{30}$, A_3, $\mathcal{F}^{B0||C}\epsilon_{BC}$ and v (contrary to (5.3) described by six functions A_k and D^k). Two of them will play the role of field configurations and the remaining two will be the conjugate momenta. Let us consider a boundary term in (5.7). Killing this term by an appropriate choice of boundary conditions is necessary for transforming the field theory into an (infinite dimensional) dynamical system (see [9], [10]). ¿From this point of view, the quantity v (or, equivalently B^3) is a good candidate for the field configuration, since controlling it at the boundary will kill the term δv in the boundary integral. On the contrary, δA_0 can not be killed by any simple boundary condition imposed on A_3. We conclude, that it is rather $\mathcal{F}^{03} = \lambda D^3$ than A_3, which has to be chosen as another field configuration.

Hence, we perform the Legendre transformation in formula (5.7) on the boundary ∂V: $\mathcal{F}^{03}\delta A_0 = \delta(\mathcal{F}^{03}A_0) - A_0\delta\mathcal{F}^{03}$. This way, using (5.10) and (5.12), we obtain from (5.7) the following result:

$$\int_V \lambda r^2 \left[-\dot{D}^3\delta(\triangle^{-1}(B^{A||B}\epsilon_{AB})) + \triangle^{-1}(\dot{B}^{A||B}\epsilon_{AB})\delta D^3 \right.$$
$$+ \left. \dot{B}^3\delta(\triangle^{-1}(D^{A||B}\epsilon_{AB})) - \triangle^{-1}(\dot{D}^{A||B}\epsilon_{AB})\delta B^3 \right]$$
$$= \delta\left(H_V^{can} - \int_{\partial V}\mathcal{F}^{03}A_0\right) + \int_{\partial V} -\lambda A_0\delta D^3 - r^2\triangle^{-1}(\mathcal{F}^{3A||B}\epsilon_{AB})\delta B^3\,, \tag{5.15}$$

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where we used the fact that the operator $\triangle^{-1}$ is self-adjoint on the functional space of monopole-free functions on a sphere.

We see that (D^3, B^3) play the role of field configurations, whereas the remaining functions $(B^{A||B}\epsilon_{AB}, D^{A||B}\epsilon_{AB})$ describe the conjugate momenta. Controlling the configurations at the boundary we kill the surface integral over ∂V and obtain this way an infinite dimensional Hamiltonian system describing the field evolution. There is, however, a problem with such a control, because the electric induction D^3 cannot be controlled freely on the boundary. The reason is that the total electric flux through both components of ∂V (i. e. through $S(r_0)$ and through the sphere at infinity) must be the same:

$$\int_{S(r_0)} \mathcal{F}^{03} = \int_{S(r_\infty)} \mathcal{F}^{03} = e\,, \tag{5.16}$$

where e is the electric charge contained in $S(r_0)$. Hence, we have to separate the monopole-free ("radiative") part of D^3 (which can be independently controlled on both ends of V) from the information about the electric charge. For this purpose we split the electric induction D^3 into

$$D^3 = \frac{e}{4\pi r^2} + \overline{D}^3\,, \tag{5.17}$$

where $\overline{D}^3$ is again a monopole-free function. It follows from (5.16) that the monopole part of D^3 (equal to $e/4\pi r^2$) is nondynamical and drops out from the volume integral of (5.4) because it is multiplied by a monopole-free function $B^{A||B}\epsilon_{AB}$. The remaining part $\overline{D}^3$ (which does not carry any information about the charge e), together with B^3, can be taken as the true, unconstrained degrees of freedom of the electromagnetic field.

In the same way we split the scalar potential A_0:

$$A_0 = \phi + \overline{A}_0\,, \tag{5.18}$$

where $\phi(r)$ is the mean value of A_0 on the sphere $S(r)$ (monopole part) and $\overline{A}_0$ is a monopole–free function ("radiative" part of A_0). Now, the boundary term $A_0\delta D^3$ in (5.4) reads

$$\int_{\partial V} \lambda A_0 \delta D^3 = \frac{1}{4\pi}\int_{\partial V} \lambda r^{-2}\phi\delta e + \int_{\partial V} \lambda \overline{A}_0 \delta \overline{D}^3\,. \tag{5.19}$$

Finally, we perform the Legendre transformation between ϕ and the monopole part of D^3 at infinity. Hence, we control the total charge contained in $S(r_0)$ and the monopole function ϕ at infinity. Since the latter does not contain any physical information and is used only to fix the residual gauge, we may use the simplest possible choice: $\phi(\infty) \equiv 0$. This way we have proved the following

THEOREM 3 *The quantities*

$$\begin{aligned}
\Psi^1 &= rB^3\,, \\
\Psi^2 &= r\overline{D}^3\,, \\
\chi_1 &= -r\triangle^{-1}(D^{A||B}\epsilon_{AB})\,, \\
\chi_2 &= r\triangle^{-1}(B^{A||B}\epsilon_{AB})
\end{aligned}$$

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together with the value e of the electric charge contained in $S(r_0)$ contain the entire (gauge invariant) information about the electromagnetic field.

Quantities Ψ^A play the role of field configurations and χ_A are conjugated momenta. Finally, formula (5.15) reads:

$$\int_V \lambda(\dot{\chi}_A \delta \Psi^A - \dot{\Psi}^A \delta \chi_A) = -\delta H_V + \int_{\partial V} \lambda \chi_A^r \delta \Psi^A + e \delta \phi(\infty) + \phi(r_0) \delta e \qquad (5.20)$$

where the new Hamiltonian H_V equals

$$H_V = H_V^{can} - \int_{\partial V} \mathcal{F}^{03} A_0 \ , \qquad (5.21)$$

and the "boundary momenta" are given by:

$$\begin{aligned} \chi_1^r &= -r \triangle^{-1}(\frac{1}{\lambda}\mathcal{F}^{3A||B}\epsilon_{AB}) \, , \\ \chi_2^r &= -r^{-1}\overline{A}_0 \, . \end{aligned}$$

They describe the response of the system to the control of the boundary values of configurations Ψ^A. The above Hamiltonian corresponds to the symmetric energy-momentum tensor of the Maxwell field (cf. [10], [11]), i.e. H_V equals numerically to the amount of the electromagnetic energy contained in the volume V. Obviously, the limit $\lim_{r_0 \to 0} H_V$ is not well defined due to the Coulomb field singularity. However, renormalizing it one gets exactly (up to a sign) the renormalized Lagrango-Hamiltonian L_H.

The space of the electromagnetic field contained in V

$$\mathcal{P}_{r_0}^{field} = \{\Psi^A, \chi_A : V \to \mathbf{R} \mid \text{with boundary conditions } \Psi^A|\partial V\}$$

is endowed with the canonical symplectic 2-form

$$\Omega_{r_0}^{field} := \int_V \lambda \delta \chi_A \wedge \delta \Psi^A \ . \qquad (5.22)$$

It may be easily obtained (see [10]) by the reduction of the standard presymplectic form $\int_V \delta \mathcal{F}^{k0} \wedge \delta A_k$ with respect to the Hamiltonian constraint $\partial_k \mathcal{F}^{k0} = 0$. The form $\Omega_{r_0}^{field}$ defines in the space of physical observables, i.e. functionals over $\mathcal{P}_{r_0}^{field}$, the canonical Poisson bracket structure:

$$\{\mathcal{F}, \mathcal{G}\}_{r_0}^{field} := \int_V \lambda \left(\frac{\delta \mathcal{F}}{\delta \Psi^A(\mathbf{x})} \frac{\delta \mathcal{G}}{\delta \chi_A(\mathbf{x})} - \frac{\delta \mathcal{G}}{\delta \Psi^A(\mathbf{x})} \frac{\delta \mathcal{F}}{\delta \chi_A(\mathbf{x})} \right) \ . \qquad (5.23)$$

6 Reduction

In this Section we finally find the space of Hamiltonian variables and the total Hamiltonian for electrodynamics of a point particle. We start with the renormalized Lagrango-Hamiltonian. It generates the Hamiltonian dynamics in the "field sector" and Lagrangian

dynamics in the "particle's sector". Our aim is to perform complete "Hamiltonization" of the theory, i.e. to perform the Legendre transformation in the particle's variables.

The original phase space is a direct sum of a particle's space $\mathcal{P}^{particle}$ and a phase space of the Maxwell field $\mathcal{P}^{field}$:

$$\mathcal{P} = \mathcal{P}^{particle} \oplus \mathcal{P}^{field} \ ,$$

where

$$\mathcal{P}^{field} := \lim_{r_0 \to 0} \mathcal{P}^{field}_{r_0} \ .$$

The particle's phase space $\mathcal{P}^{particle}$ is parameterized by q^k, v^k and conjugated momenta p_k, π_k and $\mathcal{P}^{field}$ by Ψ^A and conjugated momenta χ_A with the boundary condition $\Psi^A|\partial V$ (now V is an exterior of the sphere $S(r_0)$ with $r_0 \to 0$). The total phase space $\mathcal{P}$ is endowed with the canonical symplectic form Ω given by

$$\Omega := dp_k \wedge dq^k + d\pi_k \wedge dv^k + \Omega^{field} \ , \tag{6.1}$$

which generates the Poisson bracket in the space of functionals over $\mathcal{P}$:

$$\{\mathcal{F}, \mathcal{G}\} := \left(\frac{\partial \mathcal{F}}{\partial q^k} \frac{\partial \mathcal{G}}{\partial p_k} - \frac{\partial \mathcal{F}}{\partial p_k} \frac{\partial \mathcal{G}}{\partial q^k} \right) + \left(\frac{\partial \mathcal{F}}{\partial v^k} \frac{\partial \mathcal{G}}{\partial \pi_k} - \frac{\partial \mathcal{F}}{\partial \pi_k} \frac{\partial \mathcal{G}}{\partial v^k} \right) + \{\mathcal{F}, \mathcal{G}\}^{field} \ , \tag{6.2}$$

where

$$\Omega^{field} := \lim_{r_0 \to 0} \Omega^{field}_{r_0} \ , \tag{6.3}$$

$$\{\mathcal{F}, \mathcal{G}\}^{field} := \lim_{r_0 \to 0} \{\mathcal{F}, \mathcal{G}\}^{field}_{r_0} \ , \tag{6.4}$$

and $\{ \ , \ \}^{field}_{r_0}$ is given by (5.23). In the case of complete field theory the rest frame functionals: $\mathcal{H}$, $\mathcal{P}_k$, $\mathcal{R}_k$ and $\mathcal{S}_m$ form the Poincaré algebra with respect to $\{ \ , \ \}^{field}$. It is no longer true for the renormalized functionals. This does not mean that renormalized electrodynamics is incompatible with the special theory of relativity. Notice, that already in the context of inhomogeneous Maxwell theory with given point-like sources, field functionals do not form the Poincaré algebra but the theory is obviously relativistically invariant. We have shown in Section 3 that the Poincaré algebra structure is equivalent to the fact that the Legendre transformation (3.11) leads to the Hamiltonian theory with first class constraints. A theory which is subjected to the first class constraints is a gauge-type theory. Now, we expect that the particle's trajectory is no longer a gauge parameter but plays a dynamical role. Therefore, the "breaking" of the Poincaré algebra structure is the necessary condition for the nontrivial particle's dynamics.

To effectively reduced $\mathcal{P}$ we shall proceed as follows: we consider particle's trajectory ζ as a limiting case of a tiny world-tube of radius r_0. Therefore, calculating Poisson brackets according to (6.2) we shall keep $r_0 > 0$ and then finally go to the limit $r_0 \to 0$.

One can easily shows that only three among nine Poincaré relations (3.18) are changed, namely:

$$\begin{aligned}
\{\mathcal{P}_k, \mathcal{H}\}^{field}_{r_0} &= -e\left(\frac{\alpha_k}{6\pi r_0} + \beta_k\right) , \\
\{\mathcal{P}_k, \mathcal{P}_l\}^{field}_{r_0} &= \epsilon_{klm}B^m(r_0) , \\
\{\mathcal{P}_k, \mathcal{R}_l\}^{field}_{r_0} &= -g_{kl}\left(\mathcal{H} - m + \frac{e^2}{6\pi r_0}\right) ,
\end{aligned}$$

(6.5)

where $B^m(r_0)$ denotes the mean value of the m-th component of $\mathbf{B}$ on the sphere $S^2(r_0)$. Functions α_k and β_k are components of the dipole parts of the functions α and β, given by decompositions:

$$\mathrm{DP}(\alpha) = \alpha_k\frac{x^k}{r} , \qquad \mathrm{DP}(\beta) = \beta_k\frac{x^k}{r} .$$

Functions α and β are r^{-1} and r^0 terms respectively in the decomposition of the radial component D^r in the vicinity of the particle (see (2.6)).

Let us perform the Legendre transformation in the particle's sector in the same way as in the Section 3. The renormalized Lagrango-Hamiltonian gives rise to the Hamiltonian theory with constraints. The primary constraints $\phi^{(1)}_k$ are given obviously by (3.19). It is easy to show that the secondary constraints $\phi^{(2)}_k$

$$\phi^{(2)}_k := \{\phi^{(1)}_k, \mathbf{H}\}_{r_0} ,$$

(6.6)

where the complete Hamiltonian $\mathbf{H}$ is given by (3.20), have the same form as in (3.21). The next step in the Dirac-Bergmann procedure is to calculate the Poisson brackets between constraints.

PROPOSITION 3

$$\begin{aligned}
\{\phi^{(1)}_k, \phi^{(1)}_l\}_{r_0} &= 0 , \\
\{\phi^{(1)}_k, \phi^{(2)}_l\}_{r_0} &= \gamma\left(m - \frac{e^2}{6\pi r_0}\right)\left(g_{kl} + \gamma^2 v_k v_l\right) , \\
\{\phi^{(2)}_k, \phi^{(2)}_l\}_{r_0} &= 2\gamma e v_{[k}\delta^i_{l]}\left(\frac{\alpha_i}{6\pi r_0} + \beta_i\right) + e\epsilon_{klm}B^m(r_0) ,
\end{aligned}$$

(6.7)

where $2A_{[k}B_{l]} := A_kB_l - A_lB_k$.

We conclude that the constraints $\phi^{(1)}_k$ and $\phi^{(2)}_k$ are the same as in the case of fundamental theory, however, the constraint algebra is completely different.

DEFINITION 2 *A functional $\mathcal{F}$ over $\mathcal{P}$ is said to be first-class if its Poisson bracket with every constraint vanishes weakly.*

Observe that due to (6.7) there is no functional among $\phi_k^{(a)}$ which is first-class. The absence of the first-class constraints means that there are no gauge parameters in the theory. The constraints which are not first-class are called *second-class*. We conclude that the renormalized Lagrango-Hamiltonian gives rise to the Hamiltonian theory with second-class constraints $\phi_k^{(a)}$.

Let $\overline{\mathcal{P}}$ denotes the constrained submanifold of $\mathcal{P}$, i.e.

$$\overline{\mathcal{P}} := \{x \in \mathcal{P} \mid \phi_k^{(a)}(x) = 0\} \ ,$$

and let $e : \overline{\mathcal{P}} \to \mathcal{P}$ be an embedding. Due to the fact that $\phi_k^{(a)}$ are second-class constraints the pull-back $\overline{\Omega} = e^*\Omega$ of the symplectic form Ω is a non-degenerate 2-form on $\overline{\mathcal{P}}$ (this is equivalent to the fact that there is no gauge freedom at all). One may think that $(\overline{\mathcal{P}}, \overline{\Omega})$ is an adequate phase space for electrodynamics of a point particle. But it is not the whole story.

PROPOSITION 4 *In the finite-dimensional case a symplectic form defines an isomorphism between vectors and covectors (1-forms), i.e. if (P, ω) is a symplectic manifold, then ω induces a continuous linear map for any point $p \in P$: $\omega_p^\flat : T_pP \to T_p^*P$ defined by*

$$\omega_p^\flat(X) \cdot Y := \omega_p(X, Y) \ ,$$

for any $X, Y \in T_pP$. If $\dim P < \infty$, then $\omega_p^\flat$ is an isomorphism.

This theorem, however, is no longer true in the infinite-dimensional case.

PROPOSITION 5 *If $\dim P = \infty$, then in general $\omega_p^\flat$ is only injective, i.e. if $\omega_p^\flat(X, Y) = 0$ for all $Y \in T_pP$, then $X = 0$.*

See [15] for the proof.

DEFINITION 3 *If $\omega_p^\flat$ is only injective, then ω is called a weak symplectic form. If, moreover, $\omega_p^\flat$ is onto, then ω is called a strong symplectic.*

Let (P, ω) be a weak symplectic manifold and let X be a vector field on P defined on a dense subset D of P.

DEFINITION 4 *We call X a Hamiltonian vector field if there exists a functional $\mathcal{F} : D \to \mathbf{R}$ such that*

$$X \lrcorner \, \omega = -\delta \mathcal{F}$$

is satisfied on D.

Let us observe, that for a weak symplectic ω, there need not exist a vector field $X_{\mathcal{F}}$ corresponding to every given functional $\mathcal{F}$ on D. Therefore, from the physical point of view, weak structure is "too weak".

Let us come back to a manifold $\overline{\mathcal{P}}$ with a nondegerate 2-form $\overline{\Omega}$. Is $(\overline{\mathcal{P}}, \overline{\Omega})$ weak or strong? It turns out that $\overline{\Omega}$ is only a weak symplectic form on $\overline{\mathcal{P}}$ and the reason is very physical. The Hamiltonian vector field corresponding to the Hamiltonian $\mathbf{H}$ is not well defined on $\overline{\mathcal{P}}$. To simplify our considerations let us parametrize $\overline{\mathcal{P}}$ by a suitable coordinate system. The simplest parametrization is the following: field sector is still parameterized by Ψ^A and χ_A and particle's sector by $\mathbf{q}$ and $\mathbf{v}$. Momenta p_k and π_k are completely determined by constraints (3.19) and (3.21):

$$\pi_k(\mathbf{q}, \mathbf{v}, \text{fields}) = -\gamma^{-1}(\mathbf{a}^l_{\ k}\mathcal{R}_l - \boldsymbol{\omega}_{mk}S^m) , \qquad (6.8)$$

$$p_k(\mathbf{q}, \mathbf{v}, \text{fields}) = \gamma v_k\mathcal{H} + \gamma^{-2}\mathbf{a}^l_{\ k}\mathcal{P}_l . \qquad (6.9)$$

In fact π_k and p_k do not depend on $\mathbf{q}$. Notice, that momentum p_k canonically conjugated to particle's position q^k equals to the total momentum of the composed system (particle + field) in the laboratory frame. Now, using (3.20) and (6.9) the complete Hamiltonian $\mathbf{H}$ on $\overline{\mathcal{P}}$ is given by

$$\mathbf{H}(\mathbf{q}, \mathbf{v}; \text{fields}) = \gamma \left(\mathcal{H} + v^k\mathcal{P}_k\right) , \qquad (6.10)$$

and equals to the total energy of the composed system in the laboratory frame. Obviously, $\mathbf{H}$ is well defined at any point of $\overline{\mathcal{P}}$. However, the Hamiltonian vector field $X_{\mathbf{H}}$ is defined only on a subset $\mathcal{P}^*$ of $\overline{\mathcal{P}}$.

THEOREM 4 *The Hamiltonian vector field $X_{\mathbf{H}}$ is well defined if and only if the "fundamental equation" (2.8) is satisfied, i.e. $\mathcal{P}^*$ is defined by the following condition:*

$$\text{DP}(4\pi m\Psi^2 + e^2\Psi^2{}_{,3})(0) = 0 .$$

The proof of Theorem 4 is given in the Appendix B. The main result of this Section consists in the following

THEOREM 5 $(\mathcal{P}^*, \Omega^*)$ *is the strong symplectic manifold.*

The proof of this Theorem is given in the next Section where we construct the reduced Poisson bracket on $\mathcal{P}^*$. It turns out that the reduced bracket of any two well defined functionals over $\mathcal{P}^*$ is well defined throughout the reduced phase space.

Therefore, we finally take the space $(\mathcal{P}^*, \Omega^*)$, where Ω^* is a reduction of $\overline{\Omega}$ to $\mathcal{P}^*$, as a phase space for the composed system. Due to Theorem 5 the Hamiltonian structure $(\mathcal{P}^*, \Omega^*, \mathbf{H})$ for electrodynamics of a point particle is well defined: each state in $\mathcal{P}^*$ uniquely determines the entire history of the system. Observe, that due to (6.8) and (6.9) the reduced symplectic form possesses highly nontrivial form:

$$\Omega^* = \frac{\partial p_k}{\partial v^l}dv^l \wedge dq^k + \int_\Sigma \lambda\frac{\delta p_k}{\delta\Psi^A}\delta\Psi^A \wedge dq^k + \int_\Sigma \lambda\frac{\delta p_k}{\delta\chi_A}\delta\chi_A \wedge dq^k$$

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$$+ \quad \left(\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} \right) dv^l \wedge dv^k + \int_\Sigma \lambda \frac{\delta \pi_k}{\delta \Psi^A} \delta \Psi^A \wedge dv^k + \int_\Sigma \lambda \frac{\delta \pi_k}{\delta \chi_A} \delta \chi_A \wedge dv^k$$

$$+ \quad \int_\Sigma \lambda \delta \chi_A \wedge \delta \Psi^A . \tag{6.11}$$

It was shown in [6] that the "fundamental equation" is equivalent to the total momentum conservation. In the Lagrangian formulation of the theory [7] this equation is nothing else that the Euler-Lagrange equation of the variational problem. Now, in the Hamiltonian formulation, the "fundamental equation" is already present in the definition of the phase space of the system and the name "fundamental" is fully justified.

7   The Poisson bracket

In this Section we find the Poisson bracket structure for electrodynamics of a point particle, i.e. we find the reduced Poisson bracket $\{\ ,\ \}^*$ for the functionals over $\mathcal{P}^*$. It is defined in a obvious way via the symplectic form Ω^* on $\mathcal{P}^*$:

DEFINITION 5 *For any two functionals $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{P}^*$*

$$\{\mathcal{F}, \mathcal{G}\}^* := \Omega^*(X_\mathcal{F}, X_\mathcal{G}) .$$

To find the explicite form of $\{\ ,\ \}^*$ let us apply the Dirac method [12] (that is way some authors, e.g. [13] - [14], call it the Dirac bracket). As in the previous Section we find it firstly for $r_0 > 0$ and then go to the limit $r_0 \to 0$.

Using constraint algebra (6.7)-(6.8) let us define a following matrix (so-called Dirac matrix):

$$C_{ij}(r_0) := \left(\begin{array}{cc} \{\phi^{(1)}, \phi^{(1)}\}_{r_0} & \{\phi^{(1)}, \phi^{(2)}\}_{r_0} \\ \{\phi^{(2)}, \phi^{(1)}\}_{r_0} & \{\phi^{(2)}, \phi^{(2)}\}_{r_0} \end{array} \right)_{ij} = \left(\begin{array}{cc} 0 & X(r_0) \\ -X(r_0) & Y(r_0) \end{array} \right)_{ij} .$$

Notice, that on $\mathcal{P}^*$, i.e. on a dense subset of $\overline{\mathcal{P}}$ where the "fundamental equation" is satisfied, the constraint algebra reduces to

$$X_{ij}(r_0) = \gamma \left(m - \frac{e^2}{6\pi r_0} \right) \left(g_{ij} + \gamma^2 v_i v_j \right) , \tag{7.1}$$

$$Y_{ij}(r_0) = 2 \frac{e}{m} \gamma v_{[i} \beta_{j]} \left(m - \frac{e^2}{6\pi r_0} \right) + e \epsilon_{ijm} B^m(r_0) . \tag{7.2}$$

For a Hamiltonian theory with second-class constraints the Dirac matrix is non-singular and the inverse to $C_{ij}(r_0)$ reads:

$$C^{-1}(r_0) := \left(\begin{array}{cc} B(r_0) & -A(r_0) \\ A(r_0) & 0 \end{array} \right) ,$$

where

$$A^{ij}(r_0) \; := \; (X^{-1}(r_0))^{ij} = \frac{\gamma^{-1}(g^{ij} - v^i v^j)}{m - \frac{1}{6\pi}\frac{e^2}{r_0}} \; , \tag{7.3}$$

$$B^{ij}(r_0) \; := \; (X^{-1}(r_0)Y(r_0)X(r_0)^{-1})^{ij} =$$

$$= \; \frac{e}{m}\frac{\gamma^{-3}v^{[i}\beta^{j]}}{m - \frac{1}{6\pi}\frac{e^2}{r_0}} + e\gamma^2 B^m(r_0)\frac{\epsilon_m{}^{ij} - 2v^k v^{[i}\epsilon^{j]}{}_{km}}{(m - \frac{1}{6\pi}\frac{e^2}{r_0})^2} \; . \tag{7.4}$$

Finally, the "Dirac bracket" is defined as follows

$$\{\mathcal{F}, \mathcal{G}\}^* := \lim_{r_0 \to 0}\{\mathcal{F}, \mathcal{G}\}^*_{r_0} \; , \tag{7.5}$$

where

$$\{\mathcal{F}, \mathcal{G}\}^*_{r_0} \; := \; \{\mathcal{F}, \mathcal{G}\}_{r_0} + A^{ij}(r_0)\left(\{\mathcal{F}, \phi_i^{(1)}\}_{r_0}\{\phi_j^{(2)}, \mathcal{G}\}_{r_0} - \{\mathcal{F}, \phi_i^{(2)}\}_{r_0}\{\phi_j^{(1)}, \mathcal{G}\}_{r_0}\right)$$

$$- \; B^{ij}(r_0)\{\mathcal{F}, \phi_i^{(1)}\}_{r_0}\{\phi_j^{(1)}, \mathcal{G}\}_{r_0} \; . \tag{7.6}$$

We stress that the reduced Poisson bracket $\{\;,\;\}^*$ is well defined for any two well defined functionals $\mathcal{F}$ and $\mathcal{G}$ on the reduced phase space $\mathcal{P}^*$. Moreover, particle's and field degrees of freedom are kept at the same footing. To our knowledge it is the first consistent Poisson bracket structure for the theory of interacting particles and fields.

The complete set of "commutation relations" defined by $\{\;,\;\}^*$ is given in the Appendix C. Using these relations one easily prove the following

THEOREM 6 *Hamilton equations:*

$$\frac{d}{dt}\Psi^A(\mathbf{x}) \; = \; \{\Psi^A(\mathbf{x}), \mathbf{H}\}^* \; ,$$

$$\frac{d}{dt}\chi_A(\mathbf{x}) \; = \; \{\chi_A(\mathbf{x}), \mathbf{H}\}^* \; ,$$

reconstruct Maxwell equations in the co-moving frame (cf. [7]). Moreover,

$$\dot{q}^k = \{q^k, \mathbf{H}\}^* = v^k \; ,$$

and

$$\dot{v}^k = \{v^k, \mathbf{H}\}^* = \frac{e}{m}(\mathbf{a}^{-1})^k{}_l \beta_l \; ,$$

where the inverse of the operator $\mathbf{a}$ *defined in (3.6) reads*

$$(\mathbf{a}^{-1})^k{}_l := \gamma^{-2}\left(\delta^k{}_l - \gamma^{-1}\varphi v^k v_l\right) \; ,$$

gives the particle's "equation of motion":

$$ma_k = e\beta_k \; .$$

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8   The Hamiltonian

The formula (6.10) gives the quasi-local Hamiltonian for the composed "particle + field" system. It is expressed in therms of $\mathbf{q}$, $\mathbf{v}$ and field variables. However, these variables are highly noncanonical with respect to $\{\ ,\ \}^*$. Let us observe that momentum $\mathbf{p}$ and the particle's position $\mathbf{q}$ are still conjugated to each other with respect to the reduced bracket (7.5). Indeed, one easily shows that

$$\{q^i, p_j\}^* = \delta^i{}_j \ .$$

Therefore, we can use $\mathbf{p}$ instead of $\mathbf{v}$ to parametrize $\mathcal{P}^*$. However, to express the complete Hamiltonian $\mathbf{H}$ in terms of $\mathbf{p}$ one has to express $\mathbf{v}$ in terms of $\mathbf{q}$, $\mathbf{p}$ and fields. Using (6.9) one finds rather complicated formula for the particle's velocity (see Appendix D):

$$v_k = \frac{p^l(p_l - \mathcal{P}_l)\sqrt{\mathcal{H}^2 + p^k p_k - \mathcal{P}^k \mathcal{P}_k} - \mathcal{H}\,\mathcal{P}^l(p_l - \mathcal{P}_l)}{[p^l(p_l - \mathcal{P}_l)]^2 + \mathcal{H}^2(p^l - \mathcal{P}^l)(p_l - \mathcal{P}_l)}(p_k - \mathcal{P}_k) \ . \tag{8.1}$$

Now, inserting (8.1) into $\mathbf{H} = p_k v^k + \gamma^{-1}\mathcal{H}$ we obtain

$$\mathbf{H}(\mathbf{q}, \mathbf{p}; \text{fields}) = \sqrt{\mathcal{H}^2 + \mathbf{p}^2 - \boldsymbol{\mathcal{P}}^2} \ . \tag{8.2}$$

Let us observe that for a free particle, i.e. $e = 0$, $\mathcal{H} = m$ and $\mathcal{P}_k = 0$, formula (8.1) gives relativistic relation between particle's velocity and momentum:

$$v_k = \frac{p_k}{\sqrt{m^2 + \mathbf{p}^2}} \ ,$$

and formula (8.2) gives relativistic particle's energy:

$$E(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2} \ .$$

9   Poincaré algebra

Let us define the laboratory-frame Poincaré generators. They are given by the Lorentz transformation of the rest-frame generators $\mathcal{P}^\mu := (\mathcal{H}, \mathcal{P}^k)$, $\mathcal{R}_k$ and S^m. Obviously, laboratory-frame four-momentum p^μ is given by (6.9) and (6.10):

$$\begin{aligned} p^0 &:= \gamma\left(\mathcal{H} + v^k \mathcal{P}_k\right) \ , \\ p_k &:= \gamma v_k \mathcal{H} + \gamma^{-2}\mathbf{a}^l{}_k \mathcal{P}_l \ . \end{aligned}$$

The static moment r_k and the angular-momentum s^m with respect to the particle's position $\mathbf{q} = (q^k)$ are given as follows:

$$\begin{aligned} r_k &:= \gamma^3(\mathbf{a}^{-1})^l{}_k \mathcal{R}_l + \gamma\epsilon_{klm}v^l S^m + q_k p^0 \ , \\ s^m &:= \gamma^3(\mathbf{a}^{-1})^m{}_k S^k - \gamma\epsilon^{mkl}v_k \mathcal{R}_l + \epsilon^{mkl}q_k p_l \ . \end{aligned}$$

With these definitions one can prove the following

THEOREM 7 *Laboratory-frame generators p^0, p_k, r_k and s^m form with respect to the reduced Poisson bracket $\{\ ,\ \}^*$ the Poincaré algebra.*

This already proves that our formulation is perfectly consistent with the Lorentz invariance of the theory.

10   Particle in an external potential

Suppose now that the particle moves in an external (generalized) potential $U = U(\mathbf{q}, \dot{\mathbf{q}}, t)$. Then the Lagrango-Hamiltonian is given by:

$$L_H = -\gamma^{-1}\left(\mathcal{H} + a^k\mathcal{R}_k - \omega_m S^m\right) - U \ . \tag{10.1}$$

To pass into Hamiltonian description in the particle's variables one has to apply the same procedure as in the Section 6. The primary constraints $\phi^{(1)}_{\ k}$ are obviously given by (3.19) but the secondary ones read

$$\phi^{(2)}_{\ k} = -p_k + \gamma v_k \mathcal{H} + \gamma^2 \mathbf{a}^l_{\ k}\mathcal{P}_l - \frac{\partial U}{\partial \dot{q}^k} \approx 0 \ . \tag{10.2}$$

Obviously, $\phi^{(a)}_{\ k}$ for $a = 1, 2$; $k = 1, 2, 3$ are second-class constraints and the constraint algebra has the same form as (6.7) with β_k replaced by β^{total}_k given by

$$\beta^{total}_k = \beta_k + e^{-1}\gamma^3(\mathbf{a}^{-1})^i_{\ k}Q_i \ , \tag{10.3}$$

where

$$Q_i = -\frac{\partial U}{\partial q^i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}^i} \tag{10.4}$$

is a vector of the generalized force in the laboratory frame. The complete Hamiltonian $\mathbf{H}$ on the constraint manifold reads:

$$\mathbf{H} = p_k v^k + \gamma^{-1}\mathcal{H} + U = \gamma\left(\mathcal{H} + v^k\mathcal{P}_k\right) + U - v^k\frac{\partial U}{\partial \dot{q}^k} \ . \tag{10.5}$$

Obviously, the constraint manifold $\overline{\mathcal{P}}$ is endowed with a weak symplectic form $\overline{\Omega}$ and one can prove the following

THEOREM 8 *The Hamiltonian vector field $X_{\mathbf{H}}$ corresponding to (10.5) is well defined on the submanifold $\mathcal{P}^*$ defined by the following non-homogeneous boundary condition*

$$\mathrm{DP}(4\pi m\Psi^2 + e^2\Psi^2_{,3})(0) = -e\gamma^3(\mathbf{a}^{-1})^k_{\ i}Q_k\frac{x^i}{r} \ .$$

The proof is analogous to the proof of Theorem 4.

Finally, the Hamiltonian structure for the particle in an external potential is a triple $(\mathcal{P}^*, \Omega^*, \mathbf{H})$, where Ω^* is a reduction of $\overline{\Omega}$ to $\mathcal{P}^*$. Due to the Theorem 8 this structure is well defined, i.e. the initial data $(\Psi, \chi; \mathbf{q}, \mathbf{v})$ for the radiation field and for the particle uniquely determine the entire history of the system if the external potential is given.

As an example consider the particle interacting with an external electromagnetic field $f_{\mu\nu}^{ext}$. The generalized potential is given by:

$$U(\mathbf{q}, \dot{\mathbf{q}}, t) = eA_0^{ext}(\mathbf{q}, t) - e\dot{\mathbf{q}}\mathbf{A}^{ext}(\mathbf{q}, t) ,$$

where A_0^{ext} and $\mathbf{A}^{ext}$ stand for the four-potential of the external field in the laboratory frame. The generalized force (10.4) in terms of the laboratory-frame components $\mathcal{E}_i$ and $\mathcal{B}_i$ of the external field now reads:

$$Q_i = e \left(\mathcal{E}_i(\mathbf{q}, t) + \epsilon_{ijk} v^j \mathcal{B}^k(\mathbf{q}, t) \right) .$$

In this case

$$U - v^k \frac{\partial U}{\partial \dot{q}^k} = eA_0^{ext}$$

and the complete Hamiltonian reads

$$\mathbf{H}(\mathbf{q}, \mathbf{v}; \text{fields}) = \gamma \left(\mathcal{H} + v^k \mathcal{P}_k \right) + eA_0^{ext} .$$

Moreover, since U is linear in the velocity, we may easily express $\mathbf{v}$ in terms of $\mathbf{q}$, $\mathbf{p}$ and fields (radiation and external). One obtains formula analogous to (8.1) with p_k replaced by $p_k + eA_k^{ext}$ which leads to the following expression for $\mathbf{H}$:

$$\mathbf{H}(\mathbf{q}, \mathbf{p}; \text{fields}) = \sqrt{\mathcal{H}^2 + (\mathbf{p} + e\mathbf{A}^{ext})^2 - \mathcal{P}^2} + eA_0^{ext} .$$

Finally, let us observe that the "commutation relations" between particle's variables $\mathbf{q}$, $\mathbf{v}$ and field variables (Ψ, χ) have the same form as in Appendix C with β replaced by β^{total} defined in (10.3).

Appendixes

A   Canonical formalism for a 2-nd order Lagrangian theory

Consider a theory described by the 2-nd order lagrangian $L = L(q, \dot{q}, \ddot{q})$ (to simplify the notation we skip the index "i" corresponding to different degrees of freedom q^i; extension of this approach to higher order Lagrangians is straightforward). Introducing auxiliary variables $v = \dot{q}$ we can treat our theory as a 1-st order one with lagrangian constraints

$\phi := \dot{q} - v = 0$ on the space of lagrangian variables $(q, \dot{q}, v, \dot{v})$. Dynamics is generated by the following relation:

$$d\,L(q, v, \dot{v}) = \frac{d}{dt}(p\,dq + \pi\,dv) = \dot{p}\,dq + p\,d\dot{q} + \dot{\pi}\,dv + \pi\,d\dot{v} \,. \tag{A.1}$$

where (p, π) are momenta canonically conjugate to q and v respectively. Because L is defined only on the constraint submanifold, its derivative dL is not uniquely defined and has to be understood as a collection of *all the covectors* which are compatible with the derivative of the function along constraints. This means that the left hand side is defined up to $\mu(\dot{q} - v)$, where μ are Lagrange multipliers corresponding to constraints $\phi = 0$. We conclude that $p = \lambda$ is an arbitrary covector and (A.1) is equivalent to the system of dynamical equations:

$$\pi = \frac{\partial L}{\partial \dot{v}} \,, \qquad \dot{p} = \frac{\partial L}{\partial q} \,, \qquad \dot{\pi} = \frac{\partial L}{\partial v} - p \,.$$

The last equation implies the definition of the canonical momentum p:

$$p = \frac{\partial L}{\partial v} - \dot{\pi} = \frac{\partial L}{\partial v} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) \,. \tag{A.2}$$

We conclude, that equation

$$\dot{p} = \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \dot{v}}\right)$$

is equivalent, indeed, to the Euler-Lagrange equation:

$$\frac{\delta L}{\delta q} := \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) + \frac{\partial L}{\partial q} = 0 \,. \tag{A.3}$$

The Hamiltonian description is obtained from the Legendre transformation applied to (A.1):

$$-\,dH = \dot{p}\,dq - \dot{q}\,dp + \dot{\pi}\,dv - \dot{v}\,d\pi \,, \tag{A.4}$$

where $H(q, p, v, \pi) = p\,v + \pi\,\dot{v} - L(q, v, \dot{v})$. In this formula we have to insert $\dot{v} = \dot{v}(q, v, \pi)$, calculated from equation $\pi = \frac{\partial L}{\partial \dot{v}}$. Let us observe that H is linear with respect to the momentum p. This is a characteristic feature of the 2-nd order theory.

Euler-Lagrange equations (A.3) are of 4-th order. The corresponding 4 Hamiltonian equations have, therefore, to describe the evolution of q and its derivatives up to third order. Due to Hamiltonian equations implied by relation (A.4), the information about succesive derivatives of q is carried by (v, π, p):

- v describes $\dot{q}$

$$\dot{q} = \frac{\partial H}{\partial p} \equiv v$$

hence, the constraint $\phi = 0$ is reproduced due to linearity of H with respect to p,

- π contains information about $\dddot{q}$:

$$\dot{v} = \frac{\partial H}{\partial \pi} ,$$

- p contains information about $\ddddot{q}$

$$\dot{\pi} = -\frac{\partial H}{\partial v} = \frac{\partial L}{\partial v} - p ,$$

- the true dynamical equation equals

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{\partial L}{\partial q} .$$

B   Proof of Theorem 4

Observe, that due to (2.6) the field Ψ^2 has the following expansion in the vicinity of the particle:

$$\Psi^2(r) = \frac{\alpha}{4\pi} + r\beta + O(r^2) , \tag{B.1}$$

where $O(r^2)$ denotes terms vanishing for $r \to 0$ like r^2 or faster. We shall prove that the Hamiltonian vector field $X_{\mathbf{H}}$ corresponding to the Hamiltonian (6.10) is well defined if and only if the fundamental equation (2.8) is satisfied. The field $X_{\mathbf{H}}$ is defined by

$$X_{\mathbf{H}} \lrcorner \, \Omega^* = -\delta \mathbf{H} , \tag{B.2}$$

where Ω^* is given by (6.11) and $X_{\mathbf{H}}$ may be written in terms of coordinates as follows:

$$X_{\mathbf{H}} = X^i \frac{\partial}{\partial q^i} + Y^i \frac{\partial}{\partial v^i} + W^A \frac{\delta}{\delta \Psi^A} + Z_A \frac{\delta}{\delta \chi_A} . \tag{B.3}$$

Using (B.2) and (B.3) we get

$$
\begin{aligned}
-\frac{\partial \mathbf{H}}{\partial q^k} &= Y^l \left[\frac{\partial(\gamma v_k)}{\partial v^l} \mathcal{H} + \frac{\partial \mathbf{a}^n{}_k}{\partial v^l} \mathcal{P}_n \right] + W^A \left[\gamma v_k \frac{\delta \mathcal{H}}{\delta \Psi^A} + \mathbf{a}^n{}_k \frac{\delta \mathcal{P}_n}{\delta \Psi^A} \right] \\
&\quad + Z_A \left[\gamma v_k \frac{\delta \mathcal{H}}{\delta \chi_A} + \mathbf{a}^n{}_k \frac{\delta \mathcal{P}_n}{\delta \chi_A} \right] ,
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
-\frac{\partial \mathbf{H}}{\partial v^l} &= -X^k \left[\frac{\partial(\gamma v_k)}{\partial v^l} \mathcal{H} + \frac{\partial \mathbf{a}^n{}_k}{\partial v^l} \mathcal{P}_n \right] \\
&\quad + Y^k \left[\left(\frac{\partial(\gamma \mathbf{a}^n{}_k)}{\partial v^l} - \frac{\partial(\gamma \mathbf{a}^n{}_l)}{\partial v^k} \right) \mathcal{R}_n + \left(\frac{\partial \boldsymbol{\omega}_{km}}{\partial v^l} - \frac{\partial \boldsymbol{\omega}_{lm}}{\partial v^k} \right) S^m \right] \\
&\quad - W^A \left[\gamma \mathbf{a}^n{}_l \frac{\delta \mathcal{R}_n}{\delta \Psi^A} + \boldsymbol{\omega}_{lm} \frac{\delta S^m}{\delta \Psi^A} \right] - Z_A \left[\gamma \mathbf{a}^n{}_l \frac{\delta \mathcal{R}_n}{\delta \chi_A} + \boldsymbol{\omega}_{lm} \frac{\delta S^m}{\delta \chi_A} \right] ,
\end{aligned}
\tag{B.5}
$$

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$$-\frac{\delta \mathbf{H}}{\delta \Psi^A} = -X^k \left[\gamma v_k \frac{\delta \mathcal{H}}{\delta \Psi^A} + \mathbf{a}^n_{\ k} \frac{\delta \mathcal{P}_n}{\delta \Psi^A} \right] + Y^k \left[\gamma \mathbf{a}^n_{\ k} \frac{\delta \mathcal{R}_n}{\delta \Psi^A} + \boldsymbol{\omega}_{km} \frac{\delta S^m}{\delta \Psi^A} \right] + Z_A \ , \quad \text{(B.6)}$$

$$-\frac{\delta \mathbf{H}}{\delta \chi_A} = -X^k \left[\gamma v_k \frac{\delta \mathcal{H}}{\delta \chi_A} + \mathbf{a}^n_{\ k} \frac{\delta \mathcal{P}_n}{\delta \chi_A} \right] + Y^k \left[\gamma \mathbf{a}^n_{\ k} \frac{\delta \mathcal{R}_n}{\delta \chi_A} + \boldsymbol{\omega}_{km} \frac{\delta S^m}{\delta \chi_A} \right] - W^A \ . \quad \text{(B.7)}$$

Using the definition of $\mathbf{H}$ (see (6.10)) we have

$$\frac{\partial \mathbf{H}}{\partial q^k} = 0 \ , \tag{B.8}$$

$$\frac{\partial \mathbf{H}}{\partial v^l} = \gamma^3 v_k (\mathcal{H} + v^l \mathcal{P}_l) + \gamma \mathcal{P}_k \ , \tag{B.9}$$

$$\frac{\delta \mathbf{H}}{\delta \Psi^A} = \gamma \left[\frac{\delta \mathcal{H}}{\delta \Psi^A} + v^l \frac{\delta \mathcal{P}_l}{\delta \Psi^A} \right] \ , \tag{B.10}$$

$$\frac{\delta \mathbf{H}}{\delta \chi_A} = \gamma \left[\frac{\delta \mathcal{H}}{\delta \chi_A} + v^l \frac{\delta \mathcal{P}_l}{\delta \chi_A} \right] \ . \tag{B.11}$$

Now, calculating W^A and Z_A from (B.7) and (B.6) respectively, inserting them to (B.5) and taking into account the Poincarè algebra relations, we get

$$-A_l \mathcal{H} + B_l^{\ n} \mathcal{P}_n - \gamma^2 \mathbf{a}^n_{\ l} v^k g_{nk} \left(m - \frac{e^2}{6\pi r_0} \right)$$

$$= X^k \left[C_{lk} \mathcal{H} + D_{lk}^{\ n} \mathcal{P}_n - \gamma \mathbf{a}^n_{\ l} \mathbf{a}^r_{\ k} g_{nr} \left(m - \frac{e^2}{6\pi r_0} \right) \right] + Y^k \left[E_{lk}^{\ n} \mathcal{R}_n + F_{lkm} S^m \right] \ , \text{(B.12)}$$

where the following 3-dimensional objects depending upon the velocity $\mathbf{v}$ are introduced:

$$A_l = \gamma^3 v_l - \gamma^2 \mathbf{a}^n_{\ l} v^k g_{nk} \ ,$$

$$B_l^{\ n} = -\gamma^3 v_l v^n - \gamma \delta_l^{\ n} + \gamma^2 \mathbf{a}^n_{\ l} - \boldsymbol{\omega}_{lm} v^k \epsilon^{nm}_{\ \ k} \ ,$$

$$C_{lk} = -\gamma (g_{kl} + \gamma^2 v_k v_l) + \gamma \mathbf{a}^n_{\ l} \mathbf{a}^r_{\ k} g_{nr} \ ,$$

$$D_{lk}^{\ n} = -\frac{\partial \mathbf{a}^n_{\ k}}{\partial v^l} + \gamma^2 v_k \mathbf{a}^n_{\ l} - \boldsymbol{\omega}_{lm} \mathbf{a}^i_{\ k} \epsilon_i^{\ mn} \ ,$$

$$E_{lk}^{\ n} = \left(\frac{\partial (\gamma \mathbf{a}^n_{\ k})}{\partial v^l} - \frac{\partial (\gamma \mathbf{a}^n_{\ l})}{\partial v^k} \right) + \gamma (\mathbf{a}^i_{\ l} \boldsymbol{\omega}_{km} - \mathbf{a}^i_{\ k} \boldsymbol{\omega}_{lm}) \epsilon_i^{\ mn} \ ,$$

$$F_{lkm} = \left(\frac{\partial \boldsymbol{\omega}_{km}}{\partial v^l} - \frac{\partial \boldsymbol{\omega}_{lm}}{\partial v^k} \right) + \boldsymbol{\omega}_{lr} \boldsymbol{\omega}_{kj} \epsilon^{rj}_{\ \ m} \ .$$

Using the following properties of the function $\varphi(\tau)$:

$$2\varphi(\tau) - (1 - \tau)^{-1} + \tau \varphi^2(\tau) = 0 \ ,$$

$$2\varphi'(\tau) - (1 - \tau)^{-1} \varphi(\tau) - \varphi^2(\tau) = 0 \ ,$$

and the identity

$$v^i (\epsilon_{ikl} v_m + \epsilon_{ilm} v_k + \epsilon_{imk} v_l) = \mathbf{v}^2 \epsilon_{klm} \ ,$$

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one easily shows that

$$A_l = 0 \ , \quad B_l{}^n = 0 \ , \quad C_{lk} = 0 \ , \quad D_{lk}{}^n = 0 \ , \quad E_{lk}{}^n = 0 \ , \quad F_{lkm} = 0 \ .$$

Now, taking into account that

$$\mathbf{a}^n{}_l v_n = \gamma v_l \ ,$$
$$\mathbf{a}^n{}_l \mathbf{a}^r{}_k g_{nr} = g_{lk} + \gamma^2 v_l v_k \ ,$$

we obtain finally the following equation for X^k:

$$X^k(g_{kl} + \gamma^2 v_k v_l) = \gamma^2 v_l \ .$$

The matrix $(g_{kl} + \gamma^2 v_k v_l)$ is nonsingular and its inverse equals to $(g^{kl} - v^k v^l)$. Therefore

$$X^k = \gamma^2(g^{kl} - v^k v^l)v_l = v^k \ .$$

Now, let us consider equation (B.4) and apply the same strategy as in the case of equation (B.5). Calculate W^A and Z_A from (B.7) and (B.6) respectively and insert them into (B.4). Then, inserting $X^l = v^l$ and using once more the Poincarè algebra structure one gets:

$$\gamma(v_k v^l \mathbf{a}^n{}_l - v^2 \mathbf{a}^n{}_k + \mathbf{a}^n{}_k - \gamma v^n v_k)\left(\frac{e}{6\pi r_0}\alpha_n + e\beta_n\right)$$
$$= \ Y^l\left[\gamma(g_{kl} + \gamma^2 v_k v_l)\mathcal{H} - \mathbf{a}^n{}_l \mathbf{a}^r{}_k g_{nr}\left(\mathcal{H} - m + \frac{e^2}{6\pi r_0}\right)\right] \ , \quad\quad (\text{B.13})$$

Now, observing that

$$v_k v^l \mathbf{a}^n{}_l - v^2 \mathbf{a}^n{}_k + \mathbf{a}^n{}_k - \gamma v^n v_k = \gamma^{-2}\mathbf{a}^n{}_k$$

we finally obtain equation for Y^l:

$$Y^l(g_{kl} + \gamma^2 v_k v_l)\left(\frac{e^2}{6\pi r_0} - m\right) = \gamma^{-2}\mathbf{a}^n{}_k\left(\frac{e}{6\pi r_0}\alpha_n + e\beta_n\right) \ . \quad\quad (\text{B.14})$$

The Hamiltonian vector field $X_{\mathbf{H}}$ is well defined if and only if the singular terms proportional to r_0^{-1} cansel out, i.e.

$$eY^l(g_{kl} + \gamma^2 v_k v_l) = \gamma^{-2}\mathbf{a}^n{}_k \alpha_n \ ,$$

and therefore

$$Y^l = \frac{1}{e}\gamma^{-2}(g^{lk} - v^l v^k)\mathbf{a}^n{}_k \alpha_n = \frac{1}{e}\gamma^{-2}(g^{ln} - \gamma^{-1}\varphi \, v^l v^n)\alpha_n \ .$$

When the above equation holds then from (B.14)

$$-mY^l(g_{kl} + \gamma^2 v_k v_l) = e\gamma^{-2}\mathbf{a}^n{}_k \beta_n \ ,$$

and finally

$$m\alpha_k = -e^2\beta_k \ ,$$

which ends the proof of Theorem 4.

C Commutation relations

The complete set of "commutation relations" read:

$$\{q^k, q^l\}^* = 0 ,$$
$$\{q^k, v^l\}^* = 0 ,$$
$$\{v^k, v^l\}^* = 0 ,$$
$$\{q^k, \Psi^1(\mathbf{x})\}^* = 0 ,$$
$$\{q^k, \chi_1(\mathbf{x})\}^* = 0 ,$$
$$\{q^k, \Psi^2(\mathbf{x})\}^* = 0 ,$$
$$\{q^k, \chi_2(\mathbf{x})\}^* = \frac{3}{2e}\gamma^2(\mathbf{a}^{-1})^k{}_l \frac{x^l}{|\mathbf{x}|}\delta(|\mathbf{x}|) ,$$
$$\{v^k, \Psi^1(\mathbf{x})\}^* = 0 ,$$
$$\{v^k, \chi_1(\mathbf{x})\}^* = 0 ,$$
$$\{v^k, \Psi^2(\mathbf{x})\}^* = -\gamma(\mathbf{a}^{-1})^k{}_l \frac{x^l}{|\mathbf{x}|}\delta(|\mathbf{x}|) ,$$
$$\{v^k, \chi_2(\mathbf{x})\}^* = -\frac{3}{2m}\gamma^{-3}v^{[k}\beta^{j]}\mathbf{a}^l{}_j \frac{x_l}{|\mathbf{x}|}\delta(|\mathbf{x}|) ,$$

and for field functionals:

$$\{\Psi^1(\mathbf{x}), \Psi^1(\mathbf{y})\}^* = 0 ,$$
$$\{\Psi^1(\mathbf{x}), \chi_1(\mathbf{y})\}^* = \delta^3(\mathbf{x} - \mathbf{y}) ,$$
$$\{\Psi^1(\mathbf{x}), \Psi^2(\mathbf{y})\}^* = -\frac{3}{e}\frac{\delta\mathcal{K}_l}{\delta\chi_1(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) ,$$
$$\{\Psi^1(\mathbf{x}), \chi_2(\mathbf{y})\}^* = -\frac{3}{2e}\frac{\delta\Lambda_l}{\delta\chi_1(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) ,$$
$$\{\chi_1(\mathbf{x}), \chi_1(\mathbf{y})\}^* = 0 ,$$
$$\{\chi_1(\mathbf{x}), \Psi^2(\mathbf{y})\}^* = \frac{3}{e}\frac{\delta\mathcal{K}_l}{\delta\Psi^1(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) ,$$
$$\{\chi_1(\mathbf{x}), \chi_2(\mathbf{y})\}^* = \frac{3}{2e}\frac{\delta\Lambda_l}{\delta\Psi^1(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) ,$$
$$\{\Psi^2(\mathbf{x}), \Psi^2(\mathbf{y})\}^* = \frac{3}{e}\left(\frac{\delta\mathcal{K}_l}{\delta\chi_2(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) - \frac{\delta\mathcal{K}_l}{\delta\chi_2(\mathbf{y})} \frac{x^l}{|\mathbf{x}|}\delta(|\mathbf{x}|)\right) ,$$
$$\{\chi_2(\mathbf{x}), \chi_2(\mathbf{y})\}^* = \frac{3}{2e}\left(\frac{\delta\Lambda^l}{\delta\chi_2(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) - \frac{\delta\Lambda^l}{\delta\chi_2(\mathbf{y})} \frac{x^l}{|\mathbf{x}|}\delta(|\mathbf{x}|)\right) ,$$
$$\{\Psi^2(\mathbf{x}), \chi_2(\mathbf{y})\}^* = \delta^3(\mathbf{x} - \mathbf{y}) - \frac{9m}{2e^2}\frac{x^k}{|\mathbf{x}|}\frac{y_k}{|\mathbf{y}|}\delta(|\mathbf{x}|)\delta(|\mathbf{y}|)$$
$$- \frac{3}{2e}\frac{\delta\Lambda^l}{\delta\chi_2(\mathbf{x})} \frac{y^l}{|\mathbf{y}|}\delta(|\mathbf{y}|) - \frac{3}{e}\frac{\delta\mathcal{K}_l}{\delta\Psi^2(\mathbf{y})} \frac{x^l}{|\mathbf{x}|}\delta(|\mathbf{x}|) .$$

Functionals $\mathcal{K}_l$ and Λ_l are defined as follows:

$$\mathcal{K}_i \ := \ \mathcal{R}_i + \gamma^{-1}\varphi\, v^l\, \epsilon_{lim}S^m \ - \frac{e}{4\pi}\int_\Sigma \lambda\frac{x_i}{r^3}\Psi^2 \ ,$$

$$\Lambda_i \ := \ \mathcal{P}_i + v_i\left(\mathcal{H} + \frac{e}{m}\mathcal{K}^l\beta_l\right) - \left(mv_i - \frac{e}{2\pi}\int_\Sigma \lambda\frac{x_i}{r^3}\chi_2\right) \ .$$

D   Derivation of the formula $\mathbf{v} = \mathbf{v}(\mathbf{p})$.

From (6.9) we have

$$c_k := p_k - \mathcal{P}_k = \left(\gamma\mathcal{H} + \varphi\, v^l\mathcal{P}_l\right)v_k \ . \tag{D.1}$$

Now, let $x = v^2$, $A = \mathcal{H}\sqrt{c_k c^k} = \mathcal{H}|\mathbf{c}|$, $B = c_l\mathcal{P}^l$ and $C = c_l p^l$. From (D.1) vectors $\mathbf{c} = (c_k)$ and $\mathbf{v}$ are parallel and therefore

$$v_k = \frac{|\mathbf{v}|}{|\mathbf{c}|}c_k \ . \tag{D.2}$$

Multiplying both sides of (6.9) by v^k we obtain

$$p_k v^k = \gamma(\mathcal{H} + \mathcal{P}_k v^k) \ . \tag{D.3}$$

Moreover,

$$\mathcal{P}_k v^k = \frac{|\mathbf{v}|}{|\mathbf{c}|}c^k\mathcal{P}_k = \frac{|\mathbf{v}|}{|\mathbf{c}|}B \ . \tag{D.4}$$

and

$$p_k v^k = \frac{|\mathbf{v}|}{|\mathbf{c}|}c^k p_k = \frac{|\mathbf{v}|}{|\mathbf{c}|}C \ . \tag{D.5}$$

Therefore, inserting (D.4) and (D.5) into (D.3) we obtain

$$C = A\sqrt{\frac{x}{1-x}} + B\frac{1}{\sqrt{1-x}} \ . \tag{D.6}$$

The above equation is a square equation for $\sqrt{x}$:

$$(A^2 + C^2)x + 2AB\sqrt{x} + (B^2 - C^2) = 0 \ . \tag{D.7}$$

The discriminant Δ for (D.7) equals:

$$\Delta = 4C^2(A^2 + B^2 - C^2) \ .$$

Calculating x and inserting into (D.2) we finally obtain (8.1).

Acknowledgements

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